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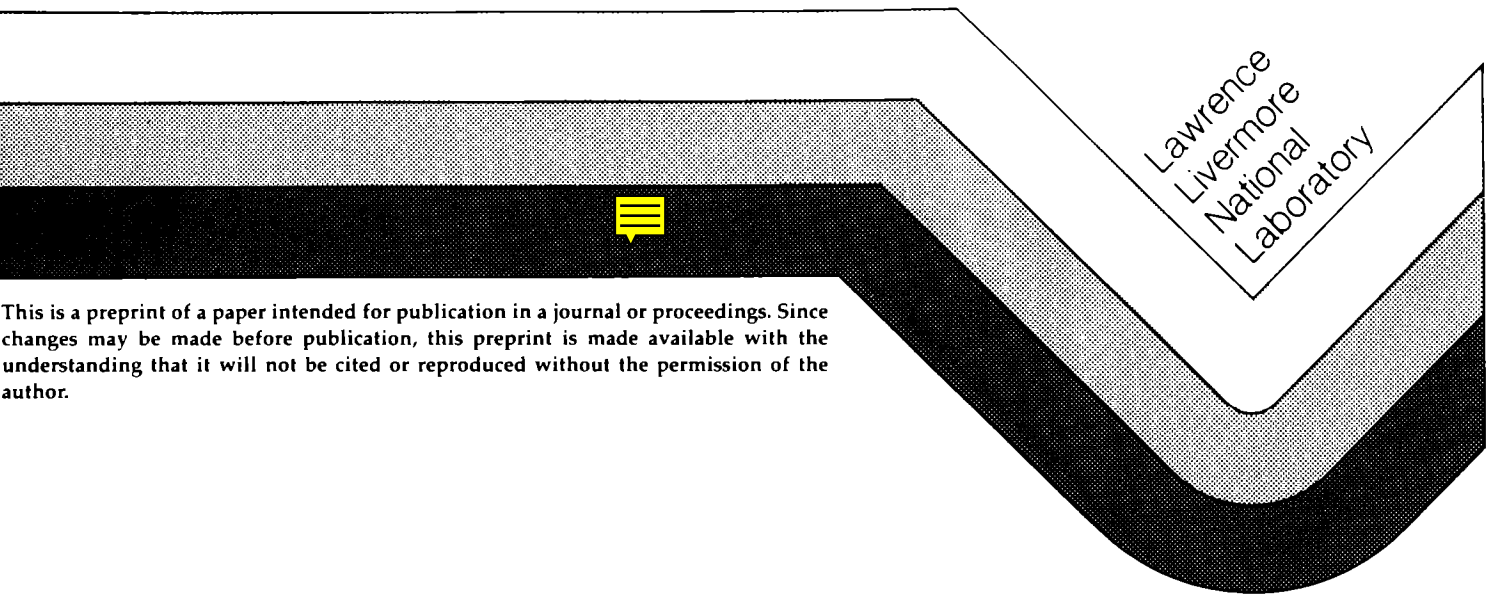

Uncertainties in predicted radionuclide-body burdens and doses from discrete stochastic source terms

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Abstract–Expressions are derived for the expectation and uncertainty of body burdens and doses calculated from a linear model of environmental transport and human metabolism in terms of expectation and uncertainty in inputs. The inputs were assumed to be discrete stochastic random variables. Three cases are compared to determine the relationship of the expectations and uncertainties under varying assumptions. In the Constant Input case, the input is selected randomly at the outset of the simulation period [0, T] from the distribution to which the population is exposed and then is held constant throughout [0, T]. In the two time-varying cases, it was assumed that N discrete stochastic exposures to the input were made; each exposure was constant during each time interval of length T/N. In one case (Random), the exposures were assumed to be uncorrelated, and in the other (Autoregressive), they were assumed to be partially correlated with autocorrelation coefficient α. The expectation values of the body burdens and doses in the Constant Input case were identical to those in the Random case. The uncertainties of the body burdens and the doses in Constant Input case were identical in the limit of Rapid Metabolism to those of the Random case. In the limit of Slow Metabolism, the uncertainties of the body burdens and the doses in the Constant Input case were $N^{1/2}$ and $(3N/4)^{1/2}$, respectively, greater than those in the Random case. If the stochastic inputs are stationary in the Autoregressive case, the expectation values of the body burden and the dose equal those for the Constant Input case. For stationary inputs, the ratio of the uncertainty of the body burden in the Autoregressive case to the uncertainty in the Constant Input case is 1 in the Rapid Metabolism limit and $\{(1+\alpha)/[(1-\alpha)N]\}^{-1/2}$ in the Slow Metabolism limit. Under the same conditions, the ratio of the uncertainty of the dose in the Rapid and Slow Metabolism limits is 1 and $\{4(1+\alpha)/[3(1-\alpha)N]\}^{-1/2}$, respectively. That is,

it is found that increasing the number of sampling periods decreases the uncertainty and increasing the autocorrelation increases the uncertainty. In an example application for ingestion of ^{137}Cs at Bikini Island, a weak form of both Slow and Rapid Metabolism limits apply and give the result that the uncertainty of the body burden in the Constant Input case is 18 time greater than the Random case. For α=0.5, the uncertainty of the body burden in the Autoregressive case is 1.7 times greater than the Random case. The smooth transition of the Autoregressive case from the Random case to the Constant Input case is shown as α increases from 0 (completely uncorrelated and random) to 1 (completely correlated and constant).

Introduction

Uncertainty in the fate and effects of radionuclides and other environmental pollutants is an important consideration in any assessment of risk. This paper considers the problem of the dependence of the output of models for computing body burdens of radionuclides and the associated dose on the uncertainty in the inputs (or forcing function). If the inputs are constant in time, this is not a difficult task because of the linear nature of the problem of exposure, body burden, and dose. However, this paper considers the case that the inputs or exposure may change over time with some uncertainty. This case requires that care must be used in the analysis.

There have been several discussions in the literature of the related problem of uncertainty in constant parameters of models for computing body burden of radionuclides (e.g., Garten 1980; Marivoet and Van Bosstraeten 1988; Breshears et al. 1989). Many authors have recommended a Monte Carlo approach to this problem, whether as a simple random design (Schwarz and Hoffman 1980; Matthies et al. 1981; O'Neill et al. 1981; Kercher and Anspaugh 1991) or as a stratified design such as Latin hypercube (Iman et al. 1981; Helton and Iman 1982; Iman and Shortencarier 1984). In the Monte Carlo approach, each parameter of the model is sampled from its distribution once before each run. By running the model many times, a distribution of the output results may be obtained. It is important to note that in most previous dose assessment schemes, each parameter

is sampled only once per run. That is, the parameter is assumed to be constant during the run. A notable exception is the work of Unnikrishnan and Prasad (1987), who considered the case of continuous random fluctuations in air activity inputs in lung-model calculations. Their analysis will be used in the calculations in a companion paper on uncertainties due to continuous stochastic inputs (Kercher 1992).

The analysis to follow was motivated by considerations of the model developed by Martin and Bloom (1980) for the Nevada Applied Ecology Group (NAEG) for application to the Nevada Test Site (NTS). For the NAEG model, Martin and Bloom assumed that a reference man as defined by the International Commission on Radiological Protection (ICRP) (ICRP 1975) was living in a desert environment contaminated by Pu (grassland-shrubland vegetation), breathing air contaminated by soil resuspension, and eating vegetables that he had grown himself in the contaminated environment and milk and beef from cattle pastured in the same contaminated environment. All concentrations of radionuclides in air and foodstuffs are modeled as proportional to the soil concentration. The soil concentration is assumed to be constant over the entire farm. That is, in the NAEG model there is an inhalation pathway from soil through resuspended particles in air to respiratory intake and there are ingestion pathways from soil to vegetables to man, from soil to pasture to milk, and from soil to pasture to meat. Ng et al. (1988) have extended the NAEG model to other radionuclides for use by BECAMP (Basic Environmental Compliance and Monitoring Program), which succeeded NAEG at the NTS. In this paper, three types of exposure to a contaminated environment are discussed. As an application of these results, the example of ingestion of ^{137}Cs in foodstuffs at Bikini Island is given after the derivations.

There are many sources of variation in the Man's intake, including day-to-day variation in ingestion of differing types of foodstuffs or of foodstuffs having within-type variation due to being grown in different parts of the farm; month-to-month variation due to differences in weather throughout the year or differences in the Man's activities at the farm; and year-to-year variations due to differences in weather, the Man's activities, or possible relocations of the Man's living site. In this paper, let us consider only discrete fluctuations, and these will be classified into three cases:

Constant Input Following Initial Random Exposure (or Constant Input for the sake of brevity), Random, and Autoregressive. The continuous case is discussed in a companion document (Kercher 1992). For the Random and Autoregressive cases, assume that the simulation period is divided into N intervals of equal duration. In the Constant Input case, the simulation period itself is one interval and undivided. In all three cases, assume that the variation within intervals is negligible. The only significant variation occurs between intervals at the transition from one interval to another. That is, the distribution of the random variable is sampled once at the beginning of each time interval. In the Random case, the soil concentrations from one interval to the next are uncorrelated. In the Autoregressive case, the soil concentration from one interval to the next have correlation coefficient α. For example, for the Man living on the farm, it might be imagined that the contamination on the farm is relatively homogeneous, but that there is regional variation that is sampled by the Man's periodic relocation. In the Constant Input case, the initial location and exposures are decided upon (sampled), and then for the duration of the simulation the exposure remains constant. The Constant Input case is the easiest to simulate.

The goals of this paper will be to compare the three cases by relating the body burdens, doses, and their uncertainties to each other. These comparisons will be made for the limits of Rapid and Slow Metabolism to simplify the algebra. However, it should be emphasized that the solutions that are given here could be used to make comparisons for arbitrary metabolism and radioactive decay rates. Closed-form solutions are given to the general problem in each of the three cases. These solutions could be used numerically to relate expectations and uncertainties between the three cases for specific radionuclides. As a secondary goal, the inputs for the Constant Input case will be determined such that the outputs reproduce the expectation and uncertainties of the Random and Autoregressive cases. These results are in Appendices C and E. For example, it will be shown how to chose the expectation and uncertainty in the soil concentration so the Constant Input case would produce the expectation and uncertainty of the body burden in the Random case. The utility of such information is that it is significantly easier to program the Constant Input case on the computer than it is to program the Random or Autoregressive cases.

For the purposes of this analysis, the uncertainty in the other model parameters will be ignored. However, in a specific assessment of a particular situation using the Monte Carlo method, one would include the uncertainty in model parameters as well as model inputs or source terms.

The basic metabolism model for man of the ICRP (ICRP 1979) calculates body burdens of a particular radionuclide with the equation

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij}\, y_j + F_i(t) \qquad (1)$$

where the a_{ij} are constant coefficients of transfers from compartment j to i (d^{-1}), y_i is the activity burden of the internal compartment i (Bq), and $F_i(t)$ is the intake (inhalation or ingestion) of the radionuclide to compartment i (Bq d^{-1}), and n is the number of compartments in the man model. The exception to this model is that of the alkaline earths (ICRP 1973) in which the a_{ij} are time-dependent. In the case of only one compartment and for $F_i(t)$ a random variable, eqn. 1 is the Langevin equation; a discussion of its subtleties can be found in standard references on stochastic processes (e.g., Wang and Uhlenbeck 1945; Prabhu 1965). In the discussion that follows, the mathematical details are simplified by assuming the matrix coefficients are constant, but suitable generalizations can be made for the time-dependent case. After eqn. 1 is solved, the cumulative internal dose for the kth organ $H_k(t)$ (Sv) is then given formally by

$$H_k(T) = \sum_{j=1}^{n} \int_{o}^{T} B_{kj}\, y_j(t)\, dt \qquad (2)$$

where the B_{kj} (Sv d^{-1} Bq $^{-1}$) are proportional to the SEE coefficients (ICRP 1979) for the kth target, jth source organ. So that one may write compact expressions, matrix notation will be used; for example eqn. 1 becomes

$$\frac{d}{dt}\mathbf{y} = \mathbf{A}'\,\mathbf{y} + \mathbf{F}(t) \tag{3}$$

where the a_{ij} are the matrix elements of **A'**, $y_i(t)$ the vector element of **y**, and $F_i(t)$ the vector elements of **F**.

The discussion in ICRP (1979) concentrates on an exposition of the **A'** matrix. Other studies concentrate on radionuclide transport in the environment and delivery to man, i.e., the forcing function **F**(t) (Hoffman et al. 1984; Whicker and Kirchner 1987; Whicker et al. 1990). Some modeling efforts model both the **A'** matrix and the **F** vector (Martin and Bloom 1980; Kercher and Anspaugh 1991). Some uncertainty analyses have concentrated on uncertainties in **A'** (Schwarz and Dunning 1982), whereas others concentrate on uncertainties in **F** (Unnikrishnan and Prasad 1987; Breshears et al. 1989). In the discussions that follow, the analysis of uncertainty is restricted to time-varying, stochastic **F**.

Development of the body burden equation

For constant matrix **A'**, the formal solution to eqn. 1 is

$$\mathbf{y}(T) = e^{\mathbf{A}T}\int_0^T e^{-\mathbf{A}t}\,\mathbf{F}(t)\,dt \;+\; e^{\mathbf{A}T}\,\mathbf{y}(0)\ . \tag{4}$$

For this discussion, consider the type of model in which the form of $F_i(t)$ is given by

$$F_i(t) = G_i(t)\ c_s(t) \tag{5}$$

where G_i is the ith component of the vector transfer function that models the transfer from the soil-to-man compartment i and $c_s(t)$ is the concentration of the radionuclide at the soil surface at time t. Models used by Martin and Bloom (1980), Kirchner et al. (1983), Kirchner and Whicker (1984),

and Kercher and Anspaugh (1991) are examples of this type. For convenience, suppose that (1) **G** is constant in time and (2) all radionuclides were deposited at time t=0, then assume that

$$c_s(t) = C_s(t)\, e^{-\lambda t} \tag{6}$$

where $C_S(t)$ is the time dependence of exposure independent of radioactive decay and λ is the radioactive decay rate of the radionuclide. Introduce a new matrix **A** equal to **A'**+ λ**I** where **I** is the identity matrix. See Appendix A for a discussion of the properties of the matrices **A** and **A'**. Also assume **y**(0) = 0. Using eqns. 5, 6, and A.3, then eqn. 4 becomes

$$\mathbf{y}(T) = e^{-\lambda T} e^{\mathbf{A}T} \int_0^T e^{-\mathbf{A}t}\ \mathbf{G}\ C_s(t)\, dt \quad . \tag{7a}$$

For some of the calculations to follow it is more convenient to write eqn. 7a as

$$y_k(T) = \sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T} \int_0^T e^{-(\lambda_j + \lambda)t}\ \Psi_{jl}^{-1}\ G_l\ C_s(t)\, dt \tag{7b}$$

where the transformation from eqn. 7a to eqn. 7b is by use of eqn. A.9. (Appendix A contains the definitions of λ_j and Ψ_{kj}).

In eqns. 7a and 7b, a man is exposed over time T to a continually changing, random soil environment contaminated by a radionuclide. That is, $C_S(t)$ is treated as a random variable. To find the uncertainty in dose due to this exposure, consider an ensemble of such men, each exposed to his environment over time T. The ensemble of men sample the contaminated environment and produce a distribution of body burdens and doses over the ensemble. The usual functions are used here to characterize such distributions and their mathematical properties are given in Appendix B. As noted above, there are uncertainties in **A** and **G** and appropriate Monte Carlo methods for analyzing those uncertainties should be used (Iman and Shortencarier 1984). In the discussion

here, let us concentrate on the uncertainties in $C_S(t)$. Let us consider three separate cases: Constant Input, Random, and Autoregressive time series. The first two cases are special cases or limits of the third, but, in the interests of clarity of expression, let us discuss each of them separately. In each case, example applications are discussed and the term *sample* is defined.

Case of Constant Input Following Initial Random Exposure

Application

The case of Constant Input Following Initial Random Exposure (or Constant Input case) corresponds to the situation in which the initial exposure is random but then stays so highly correlated during the simulation period that the exposure may be assumed to be constant. For example, Martin and Bloom (1980) assumed that the exposure to contaminated soil was constant during the simulation period (50 years). In the context of their model, this would correspond to the radionuclide being spread over the landscape as a smooth, slowly varying spatial function and once the Man chooses a location, he remains in that location and grows his food in that location for the entire time of simulation. It is assumed in this case that the normal variation to which the Man is exposed living on the farm is a small, negligible variation compared to the variation he would experience if the farm were relocated. Thus, it is assumed that the contamination is constant over the farm, but that if the farm were located elsewhere in the region there would be a different level of contamination. The randomness is for an ensemble of Men each randomly choosing a different farm in the contaminated region. In the Bikini example given below, the Constant Input case corresponds to each Marshallese ingesting constant levels of ^{137}Cs over the simulation period after initially choosing this level randomly.

Analysis

In the Constant Input case, the Man is continuously exposed to the distribution of the random variable $C_s(t)$ over the time domain of simulation $[0,T]$, but the values of the exposure are so close to the initial value that one may take as a good approximation that

$$C_s(t) = C_s(0) \; . \tag{8}$$

As a result, the most important value of $C_s(t)$ occurs at time t=0 and it ($C_s(0)$) defines the one and only *sample* of the distribution of the random variable. Each Man of an ensemble of Men chooses at time t=0 the location for his contaminated environment. The mean and standard deviation of the samples or values from the distribution of $C_s(0)$ are found over the ensemble and are denoted as c_{CI} and σ_{CI}, respectively,

$$c_{CI} = E(C_s(0)) \tag{9a}$$
$$\sigma_{CI} = D(C_s(0)) \tag{9b}$$

where E is expectation function and D the standard deviation function (for the uncertainty). These functions will be used with the usual definitions (e.g., Cramer 1955). (Appendix B provides a discussion of the properties of these statistics.) The solution to eqn. 1 becomes for constant matrix **A**'

$$\mathbf{y}(T) = e^{-\lambda T} C_s(0)\, e^{\mathbf{A}T} \int_0^T e^{-\mathbf{A}t}\, \mathbf{G}\; dt \tag{10a}$$

or

$$\mathbf{y}(T) = e^{-\lambda T} C_s(0) \left(e^{\mathbf{A}T} - \mathbf{I}\right) \mathbf{A}^{-1}\, \mathbf{G} \; . \tag{10b}$$

Using eqns. B.1 through B.6, one finds the mean and standard deviation of $y_k(t)$ for the Constant Input case to be

$$E[y_k(T)] = e^{-\lambda T} E[C_s(0)] \left[\left(e^{\mathbf{A}T} - \mathbf{I} \right) \mathbf{A}^{-1} \mathbf{G} \right]_k = e^{-\lambda T} c_{CI} \left[\left(e^{\mathbf{A}T} - \mathbf{I} \right) \mathbf{A}^{-1} \mathbf{G} \right]_k \quad (11)$$

and

$$D[y_k(T)] = e^{-\lambda T} D[C_s(0)] \left[\left(e^{\mathbf{A}T} - \mathbf{I} \right) \mathbf{A}^{-1} \mathbf{G} \right]_k = e^{-\lambda T} \sigma_{CI} \left[\left(e^{\mathbf{A}T} - \mathbf{I} \right) \mathbf{A}^{-1} \mathbf{G} \right]_k \quad (12)$$

where c_{CI} is the mean of $C_s(0)$ and σ_{CI} is the standard deviation.

Calculate the cumulative dose at time T by integrating eqn. 10b in eqn. 2. The expectation and standard deviation are found as before:

$$H_k(T) = C_s(0) \left\{ \mathbf{B}\, \mathbf{A}^{-1} \left[\left(\mathbf{A} - \lambda \mathbf{I} \right)^{-1} \left(e^{(\mathbf{A} - \lambda \mathbf{I})T} - \mathbf{I} \right) + \mathbf{I} \frac{e^{-\lambda T} - 1}{\lambda} \right] \mathbf{G} \right\}_k \quad (13)$$

$$E[H_k(T)] = c_{CI} \left\{ \mathbf{B}\, \mathbf{A}^{-1} \left[\left(\mathbf{A} - \lambda \mathbf{I} \right)^{-1} \left(e^{(\mathbf{A} - \lambda \mathbf{I})T} - \mathbf{I} \right) + \mathbf{I} \frac{e^{-\lambda T} - 1}{\lambda} \right] \mathbf{G} \right\}_k \quad (14)$$

$$D[H_k(T)] = \sigma_{CI} \left\{ \mathbf{B}\, \mathbf{A}^{-1} \left[\left(\mathbf{A} - \lambda \mathbf{I} \right)^{-1} \left(e^{(\mathbf{A} - \lambda \mathbf{I})T} - \mathbf{I} \right) + \mathbf{I} \frac{e^{-\lambda T} - 1}{\lambda} \right] \mathbf{G} \right\}_k . \quad (15)$$

Random case

Application

The idea behind both the Random and Autoregressive cases is that the inputs vary over the simulation period with a certain amount of uncertainty. In the Random case, the uncertainty from one exposure period to the next is highly uncertain, whereas in the Autoregressive case, there are correlations in inputs over the simulation period.

For both the Random and Autoregressive cases, suppose that the time period of simulation T is divided into N time intervals of length $\Delta = T/N$. Suppose that for each of the N time intervals the soil concentration $C_s(t)$ is highly correlated (constant) during that time interval but is random from one time interval to the next. That is, one assumes that for practical purposes $C_s(t)$ is

constant C_i during the ith time interval. For the Random case, in each of the N time intervals, the C_i is a random variable independent (uncorrelated) of any of the other C_i's. For example, if the spatial distribution of the radionuclide were weakly varying on the landscape and the Man were to move N times during the simulation period to new farms whose position were completely uncorrelated with those of previous farms, then this would approximate the conditions for the Random case. Each move would have to result in a new exposure to the environment independent of previous exposures. Thus, the Man could not move close to his previous position, but instead, would have to move a large, random distance away. In the Random case for the Bikini example given below, daily total ingestion of ^{137}Cs is derived from random samples of locally grown foodstuffs and a daily randomized diet based on data from a survey of Marshallese eating habits.

In the case of occupational exposure, if the Man's work assignment were to change N times over his working life with equal time intervals for each assignment, one would approximate the Random case conditions. Alternatively, if conditions on the landscape were to change at random from one small time interval to the next, say, day to day, or month to month, one might also approximate the conditions for the Random case. Seasonality is a special case of changing environmental conditions that is beyond the scope of this paper. Only nonseasonal effects will be considered here.

For the Random case, each Man in an ensemble *samples* the substrate concentration (soil for NAEG model and vegetation for Bikini application discussed below) N times during simulation length T. So, *samples* are the values of the random variables C_i. To find the mean value and the standard deviation of C_i, take the expectation and standard deviation over the ensemble. Assume that the mean and standard deviation are independent of i or the time of sampling; then the time-series of C_i is referred to as *stationary*. The mean and standard deviation are denoted

$$c_R = E(C_i) \tag{16a}$$

and

$$\sigma_R = D(C_i) \ . \tag{16b}$$

Analysis

Under assumptions of the Random case, eqn. 7a integrates to

$$\mathbf{y}(T) = e^{-\lambda T} e^{\mathbf{A}T} \left(-\mathbf{A}^{-1}\right)\left(\mathbf{I} - e^{\mathbf{A}\Delta}\right)\left[\sum_{i=1}^{N} C_i \, e^{-\mathbf{A}i\Delta}\right] \mathbf{G} \, . \tag{17}$$

Take the expectation of eqn. 17, using eqn. 16a, to get the mean of y_k, i.e.,

$$E[y_k(T)] = e^{-\lambda T} c_R \left[\mathbf{A}^{-1} \left(e^{\mathbf{A}T} - \mathbf{I}\right) \mathbf{G}\right]_k \, . \tag{18}$$

Details of the calculation are in Appendix C.

Now consider the standard deviation for the Random case. Begin with eqn. 7b and employ the Random case assumptions outlined above to get

$$D^2(y_k) = \sigma_R^2 e^{-2\lambda T} \sum_{j,l,m,r=1}^{n} \Psi_{kj} \Psi_{km} e^{-(\mu_j + \mu_m)T} \Psi_{jl}^{-1} \Psi_{mr}^{-1} G_l G_r \left\{ \frac{\left(1 - e^{-\mu_j \Delta}\right)\left(1 - e^{-\mu_m \Delta}\right)}{\mu_j \mu_m} \frac{\left(1 - e^{(\mu_j + \mu_m)T}\right)}{\left(e^{-(\mu_j + \mu_m)\Delta} - 1\right)} \right\} \, . \tag{19}$$

See Appendix C for details.

The dose calculation of the Random case is performed by integrating either eqn. 17 or C.3b in eqn. 2. to find

$$H_k(T) = \sum_{i=1}^{N} C_i \left\{ \mathbf{B}\left(-\mathbf{A}^{-1}\right)\left(-\mathbf{A} + \lambda \mathbf{I}\right)^{-1} \left[e^{-i\lambda\Delta}\left(1 - e^{\lambda\Delta}\right)\frac{\mathbf{A}}{\lambda} - \left(\mathbf{I} - e^{\mathbf{A}\Delta}\right) e^{(\mathbf{A}-\lambda)\Delta N} e^{-i\mathbf{A}\Delta} \right] \mathbf{G} \right\}_k \tag{20}$$

with

$$E[H_k(T)] = c_R \left\{ \mathbf{B}\left(-\mathbf{A}^{-1}\right)\left(-\mathbf{A} + \lambda \mathbf{I}\right)^{-1} \left[\left(1 - e^{-\lambda T}\right)\frac{-\mathbf{A}}{\lambda} - \left(\mathbf{I} - e^{\mathbf{A}T}\right) e^{-\lambda T} \right] \mathbf{G} \right\}_k \tag{21}$$

and

$$D^2(H) = \sigma_R^2 \sum_{j,l,h,p,m,q} B_{kj}\, B_{kh} \Psi_{jl}\, \Psi_{hp} \frac{\tilde{1}}{\mu_l\, \mu_p (\mu_l+\lambda)(\mu_p+\lambda)} \Psi_{lm}^{-1}\, \Psi_{pq}^{-1}\, G_m\, G_q$$

$$\left[(1-e^{\lambda\Delta})^2 \frac{\mu_l\, \mu_p}{\lambda^2} \frac{1-e^{-2\lambda\Delta N}}{e^{2\lambda\Delta}-1} + (1-e^{\lambda\Delta}) \frac{\mu_l}{\lambda} (1-e^{-\mu_p\Delta})\, e^{-(\mu_p+\lambda)\Delta N} \frac{1-e^{(\mu_p-\lambda)\Delta N}}{e^{(\lambda-\mu_p)\Delta}-1} \right. \tag{22}$$

$$+ (1-e^{\lambda\Delta}) \frac{\mu_p}{\lambda} (1-e^{-\mu_l\Delta})\, e^{-(\mu_l+\lambda)\Delta N} \frac{1-e^{(\mu_l-\lambda)\Delta N}}{e^{(\lambda-\mu_l)\Delta}-1}$$

$$\left. + (1-e^{-\mu_l\Delta})\, e^{-(\mu_l+\lambda)\Delta N} (1-e^{-\mu_p\Delta})\, e^{-(\mu_p+\lambda)\Delta N} \frac{1-e^{(\mu_l+\mu_p)\Delta N}}{e^{-(\mu_p+\mu_l)\Delta}-1} \right] .$$

Comparison of the Random and Constant Input cases

In comparing the Random and Constant Input Cases, first consider the body burdens. Note that eqn. 18 is exactly the same form for $E(y_k)$ that was determined for the Constant Input case, eqn. 11. Thus for $E(C_S(0)) = E(C_I)$ or $c_{CI} = c_R$

$$E(y_{k,\ Constant\ Input}) = E(y_{k,\ Random}) \ . \tag{23}$$

Similarly, the expectation of the dose $E(H_k)$ for the Constant Input case, eqn. 14, is exactly the same as that of the dose $E(H_k)$ for the Random case, eqn. 21, assuming that $c_{CI} = c_R$, i.e.,

$$E(H_{k,\ Constant\ Input}) = E(H_{k,\ Random}) \ . \tag{24}$$

To compare the Random case uncertainties of the body burden and the dose, eqn. 19 and 22, respectively, with the uncertainties in the Constant Input case, eqn. 12 and 15, respectively, consider these equations in two opposite and extreme limits. The first limit is for extremely rapid metabolism of the radionuclide in all of Man's compartment organs; the second limit is for extremely slow metabolism in all compartments. These limits are taken for purposes of simplifying the comparison of these uncertainties. In any particular situation regarding a particular

radionuclide, one would use the eqns. 19, 22, 12, and 15 directly to make a comparison or to determine the adjustments necessary to simulate discrete fluctuations with a constant input.

Rapid Metabolism limit

First, consider the variance of the body burdens. In the Rapid Metabolism limit, all μ_i are large, such that $\mu_i \Delta >> 1$. That is, the residence time of the radionuclide ($\tau_i = 1/\mu_i$) in each compartment i is much less than the time interval Δ. In this limit, eqn. 19 becomes

$$D(y_{k,\ Random}) \rightarrow \sigma_R\, e^{-\lambda T}\left[-\mathbf{A}^{-1}\ \mathbf{G}\right]_k \quad . \tag{25}$$

See Appendix C for details. In the same limit, eqn. 12 goes to the same expression assuming that σ_{CI} equals σ_R. So in the limit of Rapid Metabolism and for $\sigma_R = \sigma_{CI}$, the uncertainty of the body burden for the Random case $D(y_{k,Random})$ equals the uncertainty of the body burden in the Constant Input case $D(y_{k,Constant\ Input})$. In the Rapid Metabolism limit, the body burden at time T is determined by the input to the very last Δ time interval before T. Thus, even though there are N intervals, only one sampling, the last one, determines body burden. Hence the statistics of body burden for the Random case in the Rapid Metabolism limit are the same as those of the Constant Input case, which is also determined by one sample.

Next consider the variance of the dose. To compare the variance of the dose in the Constant Input case with the Random case in the Rapid Metabolism limit, let us also assume that the decay is rapid ($\lambda\Delta >> 1$). That is, the halflife of the radionuclide is much less than the time interval Δ. Then the variance of the Constant Input dose, eqn. 15, approaches

$$D(H_{k,\ Constant\ Input}) \rightarrow \frac{\sigma_{CI}}{\lambda}\left[\mathbf{B}\left(-\mathbf{A}+\lambda\,\mathbf{I}\right)^{-1}\mathbf{G}\right]_k \quad . \tag{26}$$

In this limit, the variance of the Random Dose, eqn. 22 also approaches eqn. 26 if $\sigma_R = \sigma_{CI}$. Thus, in the Rapid Metabolism limit and for σ_R equal to σ_{CI}, the uncertainty of the dose for the

Random case $D(H_{k,Random})$ equals that for the Constant Input case $D(H_{k,Constant\ Input})$. In the Rapid Metabolism and rapid decay limit, the dose at time T is determined during the very first Δ time interval. So even though there are N samples in the Random case, only one sample, the first one, is important in fixing the dose. Hence the statistics of the dose in the Random case is the same as that for the Constant Input case.

Slow Metabolism limit

Now consider the limit of Slow Metabolism. In this limit, all the μ_i are small, such that $\mu_i T << 1$. That is, the total simulation time T is much less than the residence times ($\tau_i=1/\mu_i$) of the radionuclide in each compartment i. First, examine the uncertainty in the body burdens. In Appendix C, this limit is found to be

$$D(y_{k,\ Random}) \rightarrow \frac{\sigma_R\, e^{-\lambda T}\, T}{\sqrt{N}}\, G_k \ . \tag{27}$$

In the Slow Metabolism limit, the Constant Input uncertainty (eqn. 12) becomes

$$D(y_{k,\ Constant\ Input}) \rightarrow \sigma_{CI}\, T\, e^{-\lambda T}\, G_k \ . \tag{28}$$

Thus, in the limit of Slow Metabolism and for $\sigma_R = \sigma_{CI}$

$$D(y_{k,\ Random}) = \frac{D(y_{k,\ Constant\ Input})}{\sqrt{N}} \ . \tag{29}$$

Consider next the uncertainty in the dose. In addition to the Slow Metabolism limit, let us also assume that $\lambda T << 1$. In this limit, the uncertainty of the dose in the Random case (eqn. 22) approaches

$$D(H_{k,\ Random}) \rightarrow \sigma_R [\mathbf{B}\ \mathbf{G}]_k \frac{T^2}{\sqrt{3}\,N} \ . \qquad (30)$$

In this same limit, the uncertainty of the dose for the Constant Input case, eqn. 15, approaches

$$D(H_{k,\ Constant\ Input}) \rightarrow \sigma_{CI} [\mathbf{B}\ \mathbf{G}]_k \frac{T^2}{2} \ . \qquad (31)$$

So for Slow Metabolism and decay,

$$D(H_{k,\ Random}) = \frac{\sigma_R}{\sigma_{HC}} \frac{2}{\sqrt{3}\,N} D(H_{k,\ Constant\ Input}) \ . \qquad (32)$$

Autoregressive case

Application

In the Autoregressive case, the input or source term at time t is partially dependent or correlated with the input at time $t-\Delta$ even though there may be a random component to the input. Recall that in the Random case, the input at time t is completely independent of the input at $t-\Delta$, and in the Constant Input case, the input at time t is completely dependent on the input at $t-\Delta$ with no random element. So the Autoregressive case is the transition between the Constant Input case and the Random case. Said another way, the Constant Input case and the Random case are both special cases of the Autoregressive case with the correlation either 1 or 0, respectively.

The most general linear model of random variable X for time series X_t, the Autoregressive-Moving Average (ARMA) model (Chatfield 1975; Kendall 1976), is given by

$$X_t = \alpha_1 X_{t-\Delta} + \ldots + \alpha_m X_{t-m\Delta} + \beta_0 Z_t + \beta_1 Z_{t-\Delta} + \ldots + \beta_l Z_{t-l\Delta} \qquad (33)$$

where X_t is the random variable at time $t = (i-1)\Delta$ for some integer i and Z_t is another, independent random variable at time $t = (i-1)\Delta$. The terms with the α's as coefficients make up the autoregressive part of the expression and the terms with the β's as coefficients make up the moving average part of the expression. For purposes of exposition here, the simplest autoregressive model is sufficient. So consider the case of a first-order autoregression (also known as the Markov scheme):

$$X_t = \alpha X_{t-1} + Z_t \tag{34a}$$

or for the C_i in eqns. 17 and C.3b

$$C_i = \alpha C_{i-1} + \varepsilon_i \tag{34b}$$

where ε_i is a random variable independent of C_1 and ε_j for $i \neq j$. The closed form expression for y_k will be derived for the simplest linear autocorrelation between successive samples. The derived equations will contain the Random case and Constant Input case as special cases.

The application of the Autoregressive case to the problem of exposures at the NTS as calculated by the NAEG model assumes that the soil concentration to which the Man is exposed at time t is correlated with that at t-Δ and constant over Δ. For example, if someone's work assignment required him to move around the site from week to week such that any one move resulted in changes in location less than that over which the contamination was spatially correlated, then this could lead to serial correlation of exposure and the Autoregressive case would apply. In the context of the farming assumptions of the NAEG model, examples of the Autoregressive case could be constructed due to periodic relocation of the farm, migrant workers moving from farm to farm, or movements within a large farm by one man. Construction of all these examples could be made assuming some spatial correlation of contamination and movements less than the characteristic length of the contamination. In the Bikini example given below, we only assume that there is some correlation in the day-to-day ingestion of ^{137}Cs arising either from day-to-day correlation in diet or in concentration of ^{137}Cs in food or from both.

In the Autoregressive case as in the Random case above, it is assumed that the time-series of the source-term inputs or exposures are *stationary*, meaning that the statistics of C_i are the same as those of C_j for any i and j. The results of this assumption are derived below. Finally, the *samples* for the Autoregressive case are the values of the random variable C_i, just as in the Random case.

Analysis

Note that it is well known that the autocorrelation function between C_i and C_{i+k} is α^k. In particular, the autocorrelation between neighboring time intervals is α. This is shown in Appendix C for the particular assumptions used here. Applying eqn. 34b to the C_{i-1} term in eqn. 34b successively i–2 times, one finds

$$C_i = \alpha^{i-1} C_1 + \sum_{j=0}^{i-2} \alpha^j \varepsilon_{i-j} = \alpha^{i-1} C_1 + \sum_{j=2}^{i} \alpha^{i-j} \varepsilon_j \qquad i > 1 . \qquad (35)$$

First, consider the body burdens. In the Autoregressive case, the equation for the body burden output (eqn. 17) becomes

$$\mathbf{y}(T) = e^{-\lambda T} e^{\mathbf{A}T}(-\mathbf{A}^{-1})(\mathbf{I} - e^{\mathbf{A}\Delta}) \left\{ C_1 e^{-\mathbf{A}\Delta} + \sum_{i=2}^{N} \left[\alpha^{i-1} C_1 + \sum_{j=0}^{i-2} \alpha^j \varepsilon_{i-j} \right] e^{-\mathbf{A}\Delta i} \right\} \mathbf{G} . \quad (36)$$

Calculate the expectation value of eqn. 36 using $E(C_1) = c_A$ and $E(\varepsilon_{i-j}) = \varepsilon_m$ to get

$$E(y_k) = e^{-\lambda T} \left\{ e^{\mathbf{A}T}(-\mathbf{A}^{-1})(\mathbf{I} - e^{\mathbf{A}\Delta}) e^{-\mathbf{A}\Delta} \left[c_A (\mathbf{I} - \alpha^N e^{-\mathbf{A}T})(\mathbf{I} - \alpha e^{-\mathbf{A}\Delta})^{-1} + \frac{\varepsilon_m}{1-\alpha} \left((e^{-\mathbf{A}\Delta} - e^{-\mathbf{A}T})(\mathbf{I} - e^{-\mathbf{A}\Delta})^{-1} - (\alpha e^{-\mathbf{A}\Delta} - \alpha^N e^{-\mathbf{A}T})(\mathbf{I} - \alpha e^{-\mathbf{A}\Delta})^{-1} \right) \right] \mathbf{G} \right\}_k \quad (37)$$

Note that in the limit $\alpha \to 0$, $\varepsilon_m \to c_A$, and $c_A \to c_R$, one recovers eqn. 18 for the Random case. Also, note that in the limit $\alpha \to 1$, $\varepsilon_m \to 0$, and $c_A \to c_{HC}$, one recovers eqn. 11 for the Constant Input case. More generally, for the condition

$$\varepsilon_m = c_A (1 - \alpha) \tag{38a}$$

and c_A equal to c_{CI} or c_R, eqn. 37 collapses to eqn. 11 or eqn. 18, respectively. However, for an arbitrary value of ε_m / c_A and $0 < \alpha < 1$, the expected value in the Autoregressive case differs from the Random and Constant Input cases. Equation. 38a can be derived by taking the expectation of eqn. 35 and assuming that the inputs are *stationary*, which means that the statistics of the distribution of any C_i must be equal to those of any other C_j. Likewise, by applying this principle for the D^2 function operating on eqn. 35, it is found that

$$\sigma_\varepsilon^2 = \sigma_A^2 (1 - \alpha^2) \ . \tag{38b}$$

To calculate the uncertainty in the distribution of y_k in the Autoregressive case, begin with eqn. C.3b and manipulate as shown in Appendix E to get

$$\begin{aligned} D^2(y_k) = & \left[e^{-\lambda T} e^{\mathbf{A} T} \left(\mathbf{I} - e^{-\mathbf{A} \Delta} \right) \mathbf{A}^{-1} \left(\mathbf{I} - \alpha^N e^{-\mathbf{A} T} \right) \left(\mathbf{I} - \alpha\, e^{-\mathbf{A} \Delta} \right)^{-1} \mathbf{G} \right]_k^2 \sigma_A^2 \\ & + \sigma_\varepsilon^2 \sum_{j,l,i,h=1}^{n} \Psi_{kj}\, e^{\lambda_j T}\, \Psi_{jl}^{-1}\, G_l \frac{1 - e^{-\mu_j \Delta}}{\mu_j}\, \Psi_{ki}\, e^{\lambda_i T}\, \Psi_{ih}^{-1}\, G_h \frac{1 - e^{-\mu_i \Delta}}{\mu_i} \frac{e^{(\mu_j + \mu_i)\Delta}}{\left(1 - \alpha\, e^{\mu_j \Delta}\right)\left(1 - \alpha\, e^{\mu_i \Delta}\right)} \\ & \left[\frac{e^{(\mu_j + \mu_i)\Delta} - e^{(\mu_j + \mu_i) T}}{1 - e^{(\mu_j + \mu_i)\Delta}} - e^{\mu_j T} \frac{\alpha\, e^{\mu_i T} - \alpha^N e^{\mu_i \Delta}}{e^{\mu_i \Delta} - \alpha} \right. \\ & \left. - e^{\mu_i T} \frac{\alpha\, e^{\mu_j T} - \alpha^N e^{\mu_j \Delta}}{e^{\mu_j \Delta} - \alpha} + e^{(\mu_j + \mu_i) T} \frac{\alpha^2 - \alpha^{2N}}{1 - \alpha^2} \right] \end{aligned} \tag{39}$$

Note that in the limit $\alpha \to 0$ and $\sigma_\varepsilon \to \sigma_A$, one recovers eqn. 19 for the Random case. Also, note that in the limit $\alpha \to 1$ and $\sigma_\varepsilon \to 0$, one recovers eqn. 12 for the Constant Input case. The

use of eqn. 38b automatically enforces these conditions and results in the recovery of the Random and Constant Input cases for α equal to 0 and 1, respectively. However, for an arbitrary value of $\sigma_\varepsilon/\sigma_A$ and $0 < \alpha < 1$, the expected value of D^2 in the Autoregressive case differs from the Random and Constant Input cases.

Now consider the calculation of the dose for the Autoregressive case. Integrate either eqn. 36 or E.3 in eqn. 2 to arrive at the expression for the dose to organ k

$$\begin{aligned}\mathbf{H} = \mathbf{B}\,(-\mathbf{A}^{-1})(\mathbf{A}-\lambda\,\mathbf{I})^{-1}\Bigg[\Bigg\{M(\lambda)\left(\frac{-\mathbf{A}}{\lambda}\right)\left(1-\alpha^N e^{-\lambda\Delta N}\right) + M(\mathbf{A})\,e^{(\mathbf{A}-\lambda\,\mathbf{I})\Delta N}\left(\mathbf{I}-\alpha^N e^{-\mathbf{A}\Delta N}\right)\Bigg\} C_1 \\ + M(\lambda)\left(\frac{-\mathbf{A}}{\lambda}\right) e^{\lambda\Delta}\sum_{i=2}^{N}\varepsilon_i\left(e^{-i\lambda\Delta} - \alpha^{N-i+1}e^{-\lambda\Delta(N+1)}\right) \\ + M(\mathbf{A})\,e^{(\mathbf{A}-\lambda\,\mathbf{I})\Delta N}\,e^{\mathbf{A}\Delta}\sum_{i=2}^{N}\varepsilon_i\left(e^{-i\mathbf{A}\Delta} - \alpha^{N-i+1}e^{-\mathbf{A}\Delta(N+1)}\right)\Bigg]\mathbf{G} \qquad (40a)\end{aligned}$$

where

$$M(x) = \left(1 - e^{x\Delta}\right)\left(e^{x\Delta} - \alpha\right)^{-1} . \qquad (40b)$$

The expectation of the dose is then given by

$$\begin{aligned}E(\mathbf{H}) = \mathbf{B}(-\mathbf{A}^{-1})(\mathbf{A}-\lambda\,\mathbf{I})^{-1}\Bigg[\Bigg\{M(\lambda)\left(\frac{-\mathbf{A}}{\lambda}\right)\left(1-\alpha^N e^{-\lambda\Delta N}\right) + M(\mathbf{A})\,e^{(\mathbf{A}-\lambda\,\mathbf{I})\Delta N}\left(\mathbf{I}-\alpha^N e^{-\mathbf{A}\Delta N}\right)\Bigg\} c_A \\ - \frac{\varepsilon_m}{1-\alpha}\left(\frac{-\mathbf{A}}{\lambda}\right)\left(e^{-\lambda\Delta} - e^{-N\lambda\Delta} + M(\lambda)\left(\alpha\,e^{-\lambda\Delta} - \alpha^N e^{-\lambda\Delta N}\right)\right) \\ - \frac{\varepsilon_m}{1-\alpha}\,e^{(\mathbf{A}-\lambda\,\mathbf{I})\Delta N}\left(e^{-\mathbf{A}\Delta} - e^{-N\mathbf{A}\Delta} + M(\mathbf{A})\left(\alpha\,e^{-\mathbf{A}\Delta} - \alpha^N e^{-\mathbf{A}\Delta N}\right)\right)\Bigg]\mathbf{G} \ . \qquad (41)\end{aligned}$$

To calculate the variance of the dose of the Autoregressive case, take D^2 of the eigenvalue-eigenvector form of eqn. 40a and find after manipulation

$$D^2(H_k) = \sigma_A^2 \sum_{l,j,m,q,p,r} B_{kl}\, B_{kj} \frac{\Psi_{lm}\Psi_{jq}}{\mu_m\mu_q} \left\{ M(\lambda)\left(\frac{\mu_m}{\lambda}\right) P(-\lambda) + M(-\mu_m)\, e^{-(\mu_m+\lambda)\Delta N} P(\mu_m) \right\}$$

$$\left\{ M(\lambda)\left(\frac{\mu_q}{\lambda}\right) P(-\lambda) + M(-\mu_q)\, e^{-(\mu_q+\lambda)\Delta N} P(\mu_q) \right\} \frac{\Psi_{mp}^{-1}\Psi_{qr}^{-1} G_p G_r}{(\mu_m+\lambda)(\mu_q+\lambda)}$$

$$+ \sigma_\varepsilon^2 \sum_{l,j,m,q,p,r} B_{kl}\, B_{kj} \frac{\Psi_{lm}\Psi_{jq}}{\mu_m\mu_q} \left\{ M^2(\lambda) \frac{\mu_m\mu_q}{\lambda^2} \left[R(e^{-2\lambda\Delta}) - 2\alpha^N e^{-\lambda\Delta N} R\left(\frac{e^{-\lambda\Delta}}{\alpha}\right) \right.\right.$$

$$\left. + e^{-2\lambda\Delta N}\alpha^{2N} R(\alpha^{-2}) \right] + M(\lambda)M(-\mu_q)\frac{\mu_m}{\lambda} e^{-(\mu_q+\lambda)\Delta N} \left[R(e^{(\mu_q-\lambda)\Delta}) - \alpha^N e^{-\lambda\Delta N} R\left(\frac{e^{\mu_q\Delta}}{\alpha}\right) \right.$$

$$\left. - \alpha^N e^{\mu_q\Delta N} R\left(\frac{e^{-\lambda\Delta}}{\alpha}\right) + e^{(\mu_q-\lambda)\Delta N}\alpha^{2N} R(\alpha^{-2}) \right] + M(\lambda)M(-\mu_m)\frac{\mu_q}{\lambda} e^{-(\mu_m+\lambda)\Delta N}$$

$$\left[R(e^{(\mu_m-\lambda)\Delta}) - \alpha^N e^{-\lambda\Delta N} R\left(\frac{e^{\mu_m\Delta}}{\alpha}\right) - \alpha^N e^{\mu_m\Delta N} R\left(\frac{e^{-\lambda\Delta}}{\alpha}\right) + e^{(\mu_m-\lambda)\Delta N}\alpha^{2N} R(\alpha^{-2}) \right]$$

$$+ M(-\mu_m)M(-\mu_q) e^{-(\mu_q+\mu_m+2\lambda)\Delta N} \left[R(e^{(\mu_m+\mu_q)\Delta}) - \alpha^N e^{\mu_m\Delta N} R\left(\frac{e^{\mu_q\Delta}}{\alpha}\right) - \alpha^N e^{\mu_q\Delta N} R\left(\frac{e^{\mu_m\Delta}}{\alpha}\right) \right.$$

$$\left.\left. + e^{(\mu_m+\mu_q)\Delta N}\alpha^{2N} R(\alpha^{-2}) \right] \right\} \frac{\Psi_{mp}^{-1}\Psi_{qr}^{-1} G_p G_r}{(\mu_m+\lambda)(\mu_q+\lambda)} \qquad (42a)$$

where

$$P(x) = 1 - \alpha^N e^{x\Delta N} \qquad (42b)$$

and

$$R(x) = \frac{x - x^N}{1 - x} \quad . \qquad (42c)$$

Comparison of the Autoregressive case to Constant Input case

The Random and Constant Input cases have already been compared; therefore, the Autoregressive case is compared only to the Constant Input case. To do this, compare the formulae for the Constant Input case (eqns. 11 and 12 for the body burden and eqns. 14 and 15 for the dose) with the statistics formulae for the Autoregressive case (eqns. 37 and 39 for the body burden and eqns. 41 and 42a for the dose). Note that for stationary inputs $\varepsilon_m = c_A(1-\alpha)$, the expectation of the dose in the Autoregressive case eqn. 41 becomes the same as eqns. 14 and 21, the expectations of the dose in the Constant Input case and Random case, respectively, for $c_A = c_{CI}$

and $c_A=c_R$, respectively. Because of the complexity of the eqns. 39 and 42a consider the two limits of Rapid and Slow Metabolism. Recall that a secondary goal in this discussion is to determine how to choose the mean and uncertainty of exposure for the case of Constant Input such that the output of the Constant Input case matches the mean and uncertainty of the distribution of other, more complicated cases, including the Autoregressive case. Appendix E discusses choosing inputs for the Constant Input case to do this.

Rapid Metabolism limit

In the limit of Rapid Metabolism, the expectation values of the body burden y_k for the Constant Input case and the Autoregressive case are found to be related as

$$E(y_{k,\,Autoregressive}) \rightarrow \left[\alpha^{N-1} + \frac{\varepsilon_m}{c_A(1-\alpha)}\left(1-\alpha^{N-1}\right)\right]\frac{c_A}{c_{CI}}E(y_{k,\,Constant\ Input}) \qquad (43)$$

which shows that under the stationary condition $\varepsilon_m=c_A(1-\alpha)$ and for $c_A=c_{CI}$, the mean body burden in the Autoregressive case equals the mean in the Constant Input case. Details of this calculation are in Appendix E.

Now consider the uncertainty of the body burden in the Rapid Metabolism limit. In this limit, it is shown in Appendix E that the uncertainties of the Constant Input case and the Autoregressive case are related by

$$D^2(y_{k,\,Autoregressive}) \rightarrow \frac{\sigma_A^2}{\sigma_{CI}^2}\left[\alpha^{2(N-1)} + \frac{\sigma_\varepsilon^2\left(1-\alpha^{2(N-1)}\right)}{\sigma_A^2\left(1-\alpha^2\right)}\right]D^2(y_{k,\,Constant\ Input}) \qquad (44)$$

which under the stationary condition for σ_ε^2 and for $\sigma_A=\sigma_{CI}$ insures equality between $D(y_{k,Autoregressive})$ and $D(y_{k,Constant\ Input})$.

To calculate the expectation of the dose H_k in the Rapid Metabolism limit, assume that the decay is rapid, too, i.e., let us assume $\lambda\Delta >> 1$. In Appendix E, the expectations of the dose in

the Autoregressive case and Constant Input case are calculated for this limit. They are equal for $c_A=c_{CI}$, even for nonstationary inputs.

Now consider the uncertainty in the dose for the Rapid Metabolism and rapid decay limit. In this limit, in Appendix E it is shown that the uncertainty in the Autoregressive dose is equal to the uncertainty in the Constant Input dose for $\sigma_A=\sigma_{CI}$.

Slow Metabolism limit

In the limit of Slow Metabolism for the radionuclide in question, the expectation value of the body burden y_k for Constant Input case is related to that of the Autoregressive case by

$$E(y_{k,\,Autoregressive}) \rightarrow \left[\frac{\varepsilon_m}{c_A(1-\alpha)} + \frac{1}{N}\frac{1-\alpha^N}{1-\alpha}\left(1-\frac{\varepsilon_m}{c_A(1-\alpha)}\right)\right]\frac{c_A}{c_{CI}}E(y_{k,\,Constant\ Input}) \quad (45)$$

as shown in Appendix E. Under the stationary condition for ε_m and $c_A=c_{CI}$, $E(y_{k,Autoregressive})$ is equal to $E(y_{k,Constant\ Input})$.

In the Slow Metabolism limit, one finds the uncertainty in the body burden y_k for the Constant Input case to be related to the Autoregressive case by

$$D^2(y_{k,\,Autoregressive}) \rightarrow \left[\frac{1}{N^2}\left(\frac{1-\alpha^N}{1-\alpha}\right)^2 + \frac{\sigma_\varepsilon^2}{\sigma_A^2 N^2(1-\alpha)^2}\left\{N-1-\left(\frac{1-\alpha^N}{1-\alpha^2}\right)(1-\alpha^N+2\alpha)\right\}\right] \frac{\sigma_A^2}{\sigma_{CI}^2} D^2(y_{k,\,Constant\ Input}) \quad . \quad (46)$$

See Appendix E for details. Under the conditions of stationary inputs, large N, and $\sigma_A=\sigma_{CI}$, this equation becomes

$$D^2(y_{k,\,Autoregressive}) \rightarrow \left[\frac{1}{N}\frac{1+\alpha}{1-\alpha}\right] D^2(y_{k,\,Constant\ Input}) \quad . \quad (47)$$

Note that eqns. 46 and 47 have been derived for $\alpha < 1$ such that $\mu_i T << 1-\alpha$ and $\alpha^N \rightarrow 0$; if these conditions are violated, i.e., for α very close to 1, then one must use eqn. 39 instead of eqns. 46 and 47 to calculate $D^2(y_{k,Autoregressive})$.

The expectation of the dose for the Constant Input case in the Slow Metabolism (and decay) limit is related to the Autoregressive dose by

$$E(H_{k,\ Autoregressive}) = \frac{\varepsilon_m}{c_{CI}(1-\alpha)} E(H_{k,\ Constant\ Input}) \ . \qquad (48)$$

Thus, both doses are equal under stationary conditions and $c_A = c_{CI}$.

The uncertainty of the dose for the Constant Input case in the Slow Metabolism (and decay) limit is related to that of the Autoregressive case by

$$D(H_{k\ Autoregressive}) = \frac{\sigma_\varepsilon}{\sigma_{CI}(1-\alpha)} \frac{2}{\sqrt{3N}} D(H_{k\ Constant\ Input}) \qquad (49a)$$

which under stationary conditions and for $\sigma_A = \sigma_{CI}$ becomes

$$D(H_{k,\ Autoregressive}) = \sqrt{\frac{4(1+\alpha)}{3N\ (1-\alpha)}}\ D(H_{k,\ Constant\ Input}) \ . \qquad (49b)$$

Eqns. 49a and 49b are only valid for α not near 1; for α very close to 1, then one must use eqn. 42a to calculate $D(H_{k,Autoregressive})$.

Example calculation of ingestion of ^{137}Cs at Bikini Island

The analyses in this paper were originally motivated by the need to estimate uncertainty in body burden and doses for various scenarios at the NTS as calculated by the NAEG model. At the NTS, the dominant exposure pathway is inhalation of Pu. While it is possible to treat this situation

as discrete exposures as outlined in the discussions above, situations for which the ingestion pathway dominate are more suited for the discrete exposure analysis of this paper. The NTS situation is treated as an example calculation in the companion paper on continuous exposures (Kercher 1992). At the Bikini, Enewetak, and neighboring atolls, contaminated by nuclear weapons testing, ingestion of contaminated foods is the dominant pathway and extensive data has been collected upon which realistic calculations can be made. In the Marshall Islands, the largest source of dose is ^{137}Cs (Robison and Phillips 1989), which simplifies the model of metabolism. The body burdens of ^{137}Cs for female inhabitants of Bikini Island on Bikini Atoll will be the example calculated below.

History of Bikini weapons testing and subsequent radioecological and dose-to-man investigations.

The United States conducted a nuclear test program at Bikini and Enewetak Atolls in the Marshall Islands from 1946 to 1958. Many islands at the atolls were contaminated with base-surge and close-in fallout material from the nuclear detonations. A total of 23 tests were conducted at Bikini Atoll. However, the debris cloud from the BRAVO test of 1 March 1954 at Bikini Atoll went in the opposite direction of all other tests and contaminated Bikini Island, the main residence island at Bikini Atoll, and Rongelap and Utirik Atolls to the east of Bikini.

Over the past several years, the radiological conditions at the atolls have been characterized and documented (Robison et al. 1982 a,b; 1987; 1988). The four radionuclides still present in quantities sufficient to contribute to dose calculations are ^{137}Cs, ^{90}Sr, $^{239+240}Pu$ and ^{241}Am. Robison et al. (1982b; 1987) have obtained data to evaluate all of the potential exposure pathways (terrestrial foods, marine foods, inhalation, external gamma, catchment water and ground water). The radionuclide ^{137}Cs accounts for more than 90% of the estimated dose for residents resettling the atolls and the uptake of ^{137}Cs by terrestrial foods, with subsequent ingestion by the people, contributes over 70% of the estimated dose. The external gamma exposure from the ^{137}Cs decay accounts for the remainder of the ^{137}Cs exposure. Strontium-90 is the second most significant

nuclide contributing to the estimated dose and the transuranic radionuclides contribute less than 5% of the dose over 50 and 70 y.

Consequently, in recent years, considerable effort has been spent evaluating remedial measures to reduce the uptake of ^{137}Cs into food crops to reduce the potential dose to returning inhabitants. The most effective of all the methods that have been evaluated is the application of potassium (K) to the K-deficient coral soils at the atoll. This method has proved very effective in significantly reducing the uptake of ^{137}Cs into food crops and reducing the potential dose (Robison and Stone 1992).

Because the ingestion of local foods from the contaminated islands is the major contributor to the estimated dose, the model diet employed is of obvious importance.

The basis of the model diet was the survey of the Ujelang community in 1978 by the Micronesian Legal Services Corporation (MLSC) staff and the Marshallese school teacher on Ujelang (Robison et al. 1980). Results were obtained for women, men, teenagers, and children. Adult intake exceeded that of teenagers and children, and the intake of local food was about 20% greater from women than for men. The higher intake attributed to women is unexplained, and certainly questionable. It is indicative of the acknowledged uncertainty in dietary estimates. Nevertheless, the authors suggest that the MSLC survey provides a reasonable basis for estimating dietary intake. Pending the availability of empirical data, it was decided to use the higher (female) diet as the model diet, rather than attempt further speculative refinement.

Moreover, the body burdens of ^{137}Cs predicted using the model diet and the ICRP model for ^{137}Cs in the human body (Leggett 1986; ICRP 1991 a,b) agree very closely with the actual body burdens observed by whole body counting of the people living in Rongelap and Utirik Atolls by the Brookhaven National Laboratory (Robison 1983).

^{137}Cs metabolism model

The ^{137}Cs metabolism model described in Ng et al. (1988) that is based on ICRP (1979) is used to calculate body burden. For ingestion, this model reduces to a four-compartment model

(n=4) consisting of compartments (1) stomach, (2) small intestine, (3) short-term whole body compartment, and (4) long-term whole body compartment with turnover rates μ_i of 24, 6, 0.347, and 6.30×10^{-3} d^{-1}, respectively. The off-diagonal matrix elements of **A** are $a_{21}=\mu_1$, $a_{32}=0.1\mu_2$, and $a_{42}=0.9\mu_2$. The radioactive decay rate λ of ^{137}Cs is 6.288×10^{-5} d^{-1}. The matrix elements ψ_{ij} (up to a multiplicative constant for each column) are all equal to zero except for the diagonal elements which are all equal to 1 and the off-diagonal elements ψ_{21}=-1.33, ψ_{31}=-0.02537, ψ_{41}=-0.225, ψ_{32}=-0.106, and ψ_{42}=-0.9009. Note that one could use a one compartment model for ^{137}Cs to a reasonable degree of accuracy, but the 4-compartment model is used here for expository purposes.

Ingestion

In this example, foodstuffs are the substrate in eqn. 7a, 7b, and subsequent equations rather than soil concentrations. This substitution is made because of the availability of direct information on the concentration of ^{137}Cs in foodstuffs grown on Bikini Island (Robison et al. 1988). For this exercise, Marshall Islands survey data for females between the ages of 18 and 78 for the diet consisting of both imported and locally grown food (Robison et al. 1982c) is used. Out of the total diet of 43 locally grown foodstuffs and 31 imported foodstuffs, the overwhelming ingestion of Cs is from coconut (copra meat, milk, and drinking-coconut meat and fluid) followed by *Pandanus* fruit, breadfruit, and pork. For example for Engebi Island in Enewetak Atoll, Robison et al. (1987) estimated that for 1990, 149 Bq d^{-1} would be derived from copra (meat of mature coconut) including milk (squeezed from copra), 55 from coconut fluid (free fluid in young coconuts), 39 from drinking-coconut meat, 26 from *Pandanus*, 18 from breadfruit, and 18 from pork for a total of 305 Bq d^{-1} out of a grand total of 318 Bq d^{-1} for all locally-grown foodstuffs.

To model ingestion, replace $C_s(t)$ in eqns. 7a and 7b by J

$$J(t) = \sum_{j=1}^{L} Q_j(t)\, K_j(t) \qquad (50)$$

where J is the total ingestion rate of ^{137}Cs (Bq d^{-1}), Q_j is the quantity of foodstuff j eaten (g d^{-1}), K_j is the concentration of ^{137}Cs in foodstuff j (Bq g^{-1}) and L is the number of contaminated foodstuffs. Note that the Q_j and K_j are random variables. Set G_1 in eqn. 7a and subsequent equations to 1. All other G_i are equal to 0. These assignments for **G** send all ingested material to the stomach and also keep the units consistent. For the Marshall Islands, eqn. 50 becomes

$$\begin{aligned} J = &[Q_{coconut\ juice} + 2.67\ Q_{drink.\ coc.\ meat}]\ K_{juice} + \\ &[Q_{c.\ milk} + Q_{copra} + Q_{spr.\ coc.} + Q_{cake}]\ K_{copra} \\ &+ Q_{Pandanus} K_{Pandanus} + Q_{breadfruit}\ K_{breadfruit} + Q_{pork}\ K_{pork} \end{aligned} \qquad (51)$$

where the number 2.67 is the ratio of concentration of ^{137}Cs in drinking-coconut meat to coconut juice for Bikini data. Table 2 shows the estimated mean and standard deviation of the K variables from Bikini Island (Robison et al. 1988) and the Q variables from the Marshallese diet survey.

In the example calculations in this section, four cases are shown: (1) Constant Input Following Initial Random Exposure, (2) Random, (3) Mixed Random and Constant Input, and (4) Autoregressive. It is assumed that the Δ period is 1 day and the simulation period is 5 years with t=0 being January 1, 1987, to which all data in Robison et al. (1988) has been corrected.

Constant Input Following Initial Random Exposure case

In this case, it is assumed that the ingestion rate for each foodstuff i is chosen randomly (*sampled*) at time t=0 from the diet distribution defined by the diet survey by each woman in the ensemble. Then for all days following day 1, $Q_i(t)$ is fixed at $Q_i(0)$ in eqn. 51 for each woman. Likewise, each woman randomly samples the concentration of ^{137}Cs in each foodstuff from its distribution as estimated by data from Bikini Island. This concentration is then maintained for each subsequent day, i.e., $K_i(t)$ is fixed at $K_i(0)$ in eqn. 51. The first assumption assumes that each woman has a preferred set of foodstuffs that she maintains over the course of the simulation. The second assumption is that the same plant, or similarly contaminated plants (or animals), is

continually harvested over the simulation. To complete the calculation, substitute eqn. 51 for $C_5(0)$ in eqns. 11 and 12. Take $E[J(0)]$ by using eqns. B.2 and B.5; take $D^2[J(0)]$ by using eqns. B.6 and B.7. Values for $E(K_j)$, $E(Q_j)$, $D(K_j)$, and $D(Q_j)$ for each foodstuff j are given in Table 2. Results of calculating the mean body burden $E[y_3(t)+y_4(t)]$ and uncertainty in the body burden $D[y_3(t)+y_4(t)]$ are shown in Fig. 1, as the solid line and long-dashed line, respectively.

Random case

In this case, assume that the diet for each day is chosen each day by each woman randomly (*sampled*) from the diet distribution and that the concentration of ^{137}Cs in the foodstuffs is also sampled randomly from the distributions for Bikini Island. Thus, under these assumptions, there is no correlation from day to day in either diet or location of food sources. So in eqn. 50 or 51 replace $Q_j(t)$ by $Q_j(i)$, where $Q_j(i)$ is the ingestion rate of foodstuff j on the ith day, $K_j(t)$ by $K_j(i)$, where $K_j(i)$ is the concentration of ^{137}Cs in foodstuff j on the ith day, and $J(t)$ by J_i. Substitute J_i into eqn. 19 for C_i for calculating the uncertainty of the body burdens in the Random case with σ_R the same as in the Constant Input case. The results of calculating $D[y_3(t)+y_4(t)]$ is plotted in Fig.1 as the dotted line. Note that the expectation for the Random case is the same as that calculated for the Constant Input case. While the example of ^{137}Cs ingestion at Bikini Island does not strictly fit the conditions for either the Rapid or Slow Metabolism limit stated above, weak forms of both these limits pertain to this problem. First note that the μ_4 terms dominate in eqns. 12 and 19. The elimination rate for the long-term whole body compartment (4) is such that $\mu_4\Delta << 1$ and $\mu_4 T >> 1$ for T equal to 3 to 5 years. These relations can be rewritten as $\Delta << (\tau_4=1/\mu_4) << T$ where τ_4 is the residence time of ^{137}Cs in the long-term whole body compartment. That is, the residence time is much greater than the time interval Δ and much less than the simulation time T. Under these conditions eqns. 12 and 19 have the ratio

$$\frac{D(y_{k,Random})}{D(y_{k,Constant\ Input})} \approx \frac{\sigma_R}{\sigma_{CI}}\sqrt{\frac{\Delta\mu_4}{2}} = \sqrt{\frac{\Delta}{2\tau_4}} = \sqrt{\frac{1}{2N_4}} = 0.056 \qquad (52)$$

where N_4 is the number of intervals Δ in residence time τ_4.

Case of Mixed Random and Constant Input Following Initial Random Sample

This case has two assumptions. The first is that the women have strong preferences in their diet, such that, for the consumption of food items, the Constant Input case is being followed and $Q_j(t)$ is denoted by $Q_j(0)$. The second is that when getting foodstuffs for day-to-day use, the foodstuffs are collected randomly each day anywhere on the whole island, or if food is only taken from private land then the contamination in each islander's land holding is representative for that of the whole island. That is, the conditions of the Random case are assumed to calculate the concentration of ^{137}Cs in the food items. $K_j(t)$ is denoted by $K_j(i)$ on the ith day. Substitute J_i for C_i in eqn. C.4a. Then use eqns. B.6 and B.7 to find

$$D^2(y_{k,Mixed}) = \sum_{j=1}^{L} \left[D^2(Q_j) D^2(K_j) S_k^2 + D^2(Q_j)\, E^2(K_j) W_k^2 + E^2(Q_j) D^2(K_j) S_k^2 \right] \qquad (53)$$

where S_k^2 is defined in eqn. C.4c and W_k is defined in eqn. C.2d . Using eqn. 53, the uncertainty in the Mixed case $D[y_3(t)+y_4(t)]$ is plotted in Fig. 1 as the short-dashed line.

Autoregressive case

In the Autoregressive case, it is assumed that the total intake for day i, J_i, is correlated to the previous day's total intake with a correlation coefficient α. That is, J_i is substituted for C_i in eqns. 34, 35, etc. The Autoregressive (or Partially Correlated) assumption can pertain to those situations in which the diet for each woman in the ensemble is similar from day-to-day yet has some random variation in it. Also, if there were some time-varying bias in selecting foodstuffs, then the autoregressive assumption might apply. For example, if there were contamination gradients and vegetation were harvested along the gradients over time, then the autoregressive assumption might be a good representation.

Use the stationary conditions to calculate the Autoregressive case results. Eqn. 38a assures that the mean of the body burden y_k in the Autoregressive case will be the same as the mean for the Constant Input case and for the Random case. To calculate the uncertainty in the Autoregressive case, the parameter σ_A is chosen to be the same as σ_R (or σ_{CI}). The daily autocorrelation α remains as a free parameter. In Fig. 1, the uncertainty for the body burden $D[y_3(t)+y_4(t)]$ is shown as a mixed short-long dashed line using eqn. 39. For expository purposes, a correlation of α=0.5, halfway between 0 (pure Random) and 1 (Constant Input) is chosen. Because α is a free parameter that can take on any value between 0 and 1, Fig. 2 shows the behavior of the uncertainty of the body burden in the Autoregressive case as a function of α. The uncertainty $D[y_3(t)+y_4(t)]$ is plotted at a fixed time (t=3 yr). Note that as α approaches 0, the uncertainty approaches that of the pure Random case and as α approaches 1, the uncertainty approaches that of the Constant Input case. Eqn. 39 might be applied in a dose assessment by using α derived from observations of ingestion. Another possibility for applying eqn. 39 is to find that α which optimizes the fit eqn. 39 to observed uncertainties in body burdens. As stated above, the ratio of eqn. 12 and 39 can be simplified to

$$\frac{D(y_{k,Autoregressive})}{D(y_{k,Constant\ Input})} \approx \frac{\sigma_\varepsilon}{\sigma_{CI}(1-\alpha)}\sqrt{\frac{\Delta\mu_4}{2}} = \sqrt{\frac{(1+\alpha)\Delta}{(1-\alpha)2\tau_4}} = \sqrt{\frac{(1+\alpha)}{(1-\alpha)}\frac{1}{2N_4}} = 0.097 \quad (54)$$

by applying $\mu_4\Delta << 1$ and $\mu_4 T >> 1$ for T equal to 3 to 5 years, the stationary condition $\sigma_\varepsilon^2=\sigma_A^2(1-\alpha^2)$, and $\sigma_A=\sigma_{CI}$.

Discussion

One sees by eqns. 23 and 24 that the expectation of the body burdens and doses, respectively, in the Constant Input and Random cases are equal. Recall that in the Constant Input case, every Man in an ensemble of Men samples the environment once at the beginning of a T-year exposure. In the Random case, every Man samples the environment several times in the T-year

time span with each sample being independent of all others. So, averaged over the ensemble of Men, the same average for body burden and for dose is obtained in both cases. In the Random case, the number of samples is N times that of the Constant Input case.

In comparing the Random case with the Constant Input case in the Rapid Metabolism limit, it was found above that the uncertainty in the body burdens and doses is the same in both cases, proportional to the uncertainty in the distribution of the radionuclide in the soil, and independent of the number of samples taken. In the Random case, in which many samples per simulation period are made, the Rapid Metabolism has the effect of purging one sample from the system before the end of the next time period. Thus, only the input during the last Δ interval is important in determining the body burden. In the Rapid Metabolism and rapid decay limit in the Random case, only the first Δ interval is important in determining the dose. Thus, only one sample determines the statistics of the body burden and dose. So, the statistics of the Random and Constant Input cases are the same in these limits. On the other hand, in the Slow Metabolism limit, the effect of each sampling persists for the duration of the run. In this limit, for the same uncertainty for the radionuclide concentration in the soil for both the Random and Constant Input cases, one finds that the uncertainties in the body burdens and doses in the Random case is less than those of the Constant Input case by a factor of $N^{1/2}$ and $(3N/4)^{1/2}$, respectively. So, by sampling many times, for which the content of the sampling persists in the system, the variance in the content of the system decreases by a factor of N. The dependence of the uncertainty on N in the Slow Metabolism limit is analogous to that of the standard error of the mean of N if N were to represent the number of samples used to estimate the mean. Now, if one wants to simulate the effect of the multiple samples in situations in which the environment is sampled only once at the beginning of the run, one may do so by decreasing the uncertainty in the soil distribution by $N^{1/2}$ for body burdens or $(3N/4)^{1/2}$ for doses. That is, one can use the model in the constant input mode to simulate a variable input of multiple samples if it is known that the multiple sampling occurs in N discrete, equal time intervals. Note that the factor of $3^{1/2}/2$ arises from the definition of dose as proportional to the time integral of body burden.

Now consider the Autoregressive case. The Autoregressive case corresponds to real-world situations in which each Man makes N successive samples of the environment over the course $[0,T]$ of the simulation, but for which each sample is not independent of the previous samples. Instead, each sample is correlated with the previous sample with a correlation coefficient of α, correlated with the second previous sample by α^2, etc. An example might be that the Man lives in an environment for which the radionuclide contamination is a spatial function with small spatial variation over the distances of possible moves by the Man. In this example, the function of spatial contamination would have small-scale, spatial noise associated with it. If the Man moves a small distance in this environment N times, his exposure could approximate the conditions of the Autoregressive case. Alternatively, in an occupational setting, N reassignments during a working career in which each reassignment had some similarity with the previous assignment and some new additional features could also approximate the conditions of the Autoregressive case.

The expectation of the body burdens and of the doses for the Constant Input case are equal to the expectations in the Autoregressive case if c_{CI} equals c_A and c_A equals $\varepsilon_m (1-\alpha)^{-1}$. For the Rapid Metabolism and rapid decay limit, the expectation of the dose in the Constant Input case is equal to the expectation in the Autoregressive case if c_A equals c_{CI} even under nonstationary conditions. In the Rapid Metabolism and rapid decay limit the initial sample is the most important. In the Slow Metabolism limit, the initial sample c_A in the Autoregressive case becomes increasingly unimportant as N gets large because α^N (the correlation of the last sample with c_A) is small for $\alpha < 1$. Hence, the mean of the noise term in the sampling is the important factor in determining final body burdens. But because any noise introduced in any sampling persists into further time intervals, the mean of the sampled noise is inflated by a factor of $(1-\alpha)^{-1}$.

In the Rapid Metabolism limit for large values of N, the uncertainty of the body burden in the Autoregressive case is equal to the uncertainty in the Constant Input case multiplied by a factor of $\sigma_\varepsilon \sigma_{CI}{}^{-1}(1-\alpha^2)^{-1/2}$, which becomes 1 for stationary input and for $\sigma_{CI}=\sigma_A$. One finds that σ_A, the uncertainty in the first sample of the soil concentration, is unimportant in determining the uncertainty in the body burden in the Autoregressive case. However, in the Rapid Metabolism and

rapid decay limit, the uncertainty of the dose in the Autoregressive case is related to the uncertainty of the dose in the Constant Input case by the factor σ_A/σ_{CI}. Hence, σ_A is important to the dose in this limit. Also, note that for large N, the asymptotic expressions for the uncertainty in both the body burdens and doses in the Autoregressive case in the Rapid limit is independent of N. The radionuclide activity taken up in each to the N time periods is turned over before the end of the next period. Thus, each sample's impact on the final burden due to persistence of radionuclide burden is minimal, and the uncertainty is independent of the number of samples. However, the value of the radionuclide sampled near the end of the simulation depends on previous samples because of the autocorrelation function α^k. The uncertainty in the body burden for the Autoregressive case is increased by a factor of $(1-\alpha^2)^{-1/2}$ for nonstationary input because of this reduced independence. But for stationary input, the uncertainty of the body burden in the Constant Input case matches the uncertainty of the Autoregressive case. However, in the Rapid Metabolism and rapid decay limit, the uncertainty in the dose is determined solely by the uncertainty in the initial exposure σ_A.

In the Slow Metabolism limit for large N, the uncertainty in the body burdens and doses for the Autoregressive case is equal to the uncertainty in the Constant Input case multiplied by a factor of $\sigma_\varepsilon\sigma_{CI}^{-1}[(1-\alpha)(N^{1/2})]^{-1}$ and $\sigma_\varepsilon\ \sigma_{CI}^{-1}\ [(1-\alpha)(N^{1/2})]^{-1}\ [(2/(3^{1/2})]$, respectively. For stationary input and $\sigma_{CI}=\sigma_A$, these factors become $\{(1+\alpha)/[N(1-\alpha)]\}^{-1/2}$ and $\{4(1+\alpha)/[3N(1-\alpha)]\}^{-1/2}$, respectively. Because of the Slow Metabolism, each sampled radionuclide concentration persists in the body burden. This persistence decreases the uncertainty in the final body burden by the factor of $N^{1/2}$. Again, Slow Metabolism produces a dependence on N similar to that of the standard error. The effect of the correlation of the samples from one time period to the next reduces the randomness between time intervals and increases the uncertainty in the body burden by the factor of $[(1+\alpha)/(1-\alpha)^{-1/2}$. To force the Constant Input case to match the uncertainty in the body burdens or the dose in the Autoregressive case requires that one multiplies the uncertainty in the soil concentration by $\{(1+\alpha)/(1-\alpha)N)]^{-1/2}$ or $\{4(1+a)/[3(1-\alpha)N]\}^{-1/2}$, respectively. The difference of a factor of $2/(3^{1/2})$ between these two expressions arises from the definition of dose as an integral of body burden.

The expressions derived in this paper for body burden and dose in the various cases are sufficiently complicated that it is difficult to understand them except in special limits or by graphical display in specific applications. In an example application, the mean and standard deviation in the body burden of ^{137}Cs in adult females living at Bikini Island is shown for the four cases of Constant Input Following Initial Random Exposure, Random, Mixed Random and Constant Input, and Autoregressive. The means for all four cases are the same, including the Autoregressive case for which the condition of stationary input is necessary to achieve equality. The uncertainty for the case of Constant Input is about 90% that of the mean at 3 years exposure. The uncertainties of the Mixed, Random, and Autoregressive cases are 75, 5.6, and 9.6%, respectively, of the uncertainty of the Constant Input case. The Random case uncertainty is sharply reduced because a weak form of both the Slow and Rapid Metabolism limit applies in which $\mu_4\Delta << 1$ (the residence time of ^{137}Cs in the long-term whole body compartment is much greater than Δ) and $\mu_4 T >> 1$ (the residence time is much less than T) where the fourth compartment dominates the whole-body body burden. These conditions are better written $\Delta<<1/\mu_4<<T$, which indicates that the residence time of ^{137}Cs in compartment 4 is much greater than the randomizing time-interval Δ and much less than the total simulation time T (3 to 5 years). In this limit, the reduction factors of the uncertainties are $\sim(\mu_4\Delta/2)^{1/2}$ and $\sim\{\mu_4\Delta(1+\alpha)/[2(1-\alpha)]\}^{1/2}$ for the Random and Autoregressive cases, respectively. Thus, the effect of the sampling for each Δ time period reduces the uncertainty because of mixing in the compartment of the various sampled quantities from each Δ time interval during the residence time. However, in the Autoregressive case, because of autocorrelation, the level of input tends to persist and this persistence increases the uncertainty over that of the purely Random case by a factor of $[(1+\alpha)/(1-\alpha)]^{1/2}$. The uncertainty in the Autoregressive case (eqn. 39) is plotted in Fig. 2 for all α. The uncertainty in the figure shows a gradual increase from the purely Random case (α=0) up until about α=0.9, followed by a rapid rise to the purely Constant Input case at α=1. Note that any value of the uncertainty, between that for the Random case and the Constant Input case, can be simulated by an appropriate choice of α. Thus, one could simulate phenomenologically very complicated situations of inadequately known input statistics by the

Autoregressive case with α appropriately chosen so that the statistics of the Autoregressive output match those of the complicated case.

It should be noted that the underlying shapes of the distributions have not been assumed or derived. For example, the results of Kercher and Anspaugh (1991) suggested that the NAEG model produced lognormal distributions for the body burden of Pu. Only the means and variance, or first and second moments, have been used in this paper. For complicated models of body burdens and doses, it is still very useful to use a Monte Carlo method, or one of its variations, to find the form or shape of the distribution.

Conclusion

The results of this paper indicate that the rate of metabolism has an important effect on the uncertainty in body burdens of radionuclides and doses in situations in which the exposure to the radionuclide changes over time in a stochastic way. Slow Metabolism tends to reduce uncertainty relative to those situations in which the soil concentration is sampled once and then held constant. The results also indicate that under Slow Metabolism, as serial autocorrelation of inputs or uptake rate increases, the uncertainty in body burden and dose also increases. The condition of high initial inputs tending to remain high and low initial inputs tending to remain low increases the uncertainty. For the example of ingestion of ^{137}Cs on Bikini Island, it was found that for intake chosen randomly daily based on diet and ^{137}Cs concentration distributions (Random case), the body burden was a factor of 18 times less than that for the case of intake chosen randomly at the beginning and then fixed (Constant Input). The uncertainty of the body burden in the Autoregressive case went smoothly between the uncertainties of body burden for the two extreme cases (Constant Input and Random) as the correlation coefficient a varied from 0 (Random) to 1 (Constant Input). These results suggest that complicated cases for which observed uncertainties lie between the two extreme cases can be treated phenomenologically by fitting the Autoregressive

case to the observations. While care must be used in this approach, it may prove to be a useful diagnostic tool.

If enough is known about the sampling over the simulation period, then the uncertainty distribution for the Constant Input case can be adjusted so that simulations using constant inputs can simulate the uncertainty of the body burdens or doses in the Random or Autoregressive cases. In particular, one needs to know the number of sampling periods, they must be the same for all members of the ensemble, and they must be of equal length. One needs to know or be able to estimate the degree of autocorrelation between the successive exposures. Finally, an estimate of the mean and uncertainty in the random portion of the exposure must be made.

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References

Breshears, D. D.; Kirchner, T. B.; Otis, M. D.; Whicker, F. W. Uncertainty in predictions of fallout radionuclides in foods and of subsequent ingestion. Health Phys. 57:943–953; 1989.

Chatfield, C. The analysis of time series: Theory and practice. New York: John Wiley; 1975.

Cramer, H. The elements of probability theory. New York: John Wiley; 1955.

Garten, C. T., Jr. Statistical uncertainties in predicting plutonium dose to lung and bone from contaminated soils. Health Phys. 39:99–103; 1980.

Helton, J. C.; Iman, R. L. Sensitivity analysis of a model for the environmental movement of radionuclides. Health Phys. 42:565–584; 1982.

Hoffman, F. O.; Bergstrom, U.; Gyllander, C.; Wilkens, A. B. Comparison of predictions from internationally recognized assessment models for the transfer of selected radionuclides through terrestrial food chains. Nucl. Safety 25:533–546; 1984.

Iman, R. L.; Helton, J. C.; Campbell, J. E. An approach to sensitivity analysis of computer models:Part I-introduction, input variable selection and preliminary variable assessment. J. Qual. Technol. 13:174–183; 1981.

Iman, R. L.; Shortencarier, M. J. A FORTRAN 77 program and user's guide for the generation of Latin hypercube and random samples for use with computer models. Albuquerque, NM: Sandia National Laboratories Report No. NUREG/CR-3524. SAND83-2365; 1984..

International Commission on Radiological Protection. Radiation Protection. Alkaline earth metabolism in adult man. Oxford: Pergamon Press; ICRP Publication 20; 1973.

International Commission on Radiological Protection. Report of the Task Group on Reference Man. Oxford: Pergamon Press; ICRP Publication 23; 1975.

International Commission on Radiological Protection. Limits for intakes of radionuclides by workers. Oxford: Pergamon Press; ICRP Publication 30, Part 1; Ann. ICRP 2(3/4); 1979.

International Commission on Radiological Protection. 1990 Recommendations of the International Commission on Radiological Protection. New York: Pergamon Press, Publication 60: 1991a.

International Commission on Radiological Protection. Annual limits of intake of radionuclides by workers based on the 1990 recommendations. New York: Pergamon Press, Publication 61; 1991b.

Kendall, M. Time-series. 2nd Ed. New York: Hafner Press; 1976.

Kercher, J. R. Closed-form solutions to sensitivity equations in the frequency and time domains for linear models of ecosystems. Ecol. Modelling 18:209–221; 1983.

Kercher, J. R.; Anspaugh, L. R. Analysis of the Nevada-Applied-Ecology-Group model of transuranic radionuclide transport and dose. J. Environ. Radioactivity 13:191–216; 1991.

Kercher, J. R. Continuous stochastic source terms and uncertainties in predicted radionuclide-body burdens and doses. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-JC-108805; 1992.

Kirchner, T. B.; Otis, M. D.; Whicker, F. W. PATHWAY: A simulation model of radionuclide transport through agricultural food chains. *In* : Lauenroth, W.K.; Skogerbee, GV.; Flug, M., eds. Analysis of ecological systems: State-of-the-art in ecological modeling. New York: Elsevier Scientific Publ. Co; 1983; 959–968.

Kirchner, R. B.; Whicker, F. W. Validation of PATHWAY: A simulation model for the transport of radionuclides though agroecosystems. Ecol. Mod. 22:21–44; 1984.

Leggett, R. W. Predicting the retention of cesium in individuals. Health Phys. 50: 747-759; 1986.

Marivoet, J.; Van Bosstraeten, C. Probabilistic performance assessment for radioactive waste disposal: A simplified biosphere model. Health Phys. 55:993–1000; 1988.

Martin, W. E.; Bloom, S. G. Nevada applied ecology group model for estimating plutonium transport and dose to man. *In* :Transuranic Elements in the Environment. Washington, DC: U.S. Department of Energy; Report No. DOE/TIC-22800; 1980; 459-512.

Matthies, M.; Eisfeld, K.; Paretzke, H.; Wirth, E. Stochastic calculations for radiation risk assessment: A Monte-Carlo approach to the simulation of radiocesium transport in the pasture-cow-milk food chain. Health Phys. 40:764–769; 1981.

Ng, Y.C.; Rodean, H.C.; Anspaugh, L.R. Incorporation of additional radionuclides and the external exposure pathway into the BECAMP radiological assessment model. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-53893; 1988.

O'Neill, R. V.; Gardner, R. H.; Hoffman, F.O.; Schwarz, G. Parameter uncertainty and estimated radiological dose to man from atmospheric ^{131}I releases: A Monte Carlo approach. Health Phys. 40:760–764; 1981.

Prabhu, N.U. Stochastic processes. New York: Macmillan; 1965.

Robison, W. L. Radiological dose assessments of atolls in the northern Marshall Islands. In: Environmental Radioactivity, proceedings of nineteenth annual meeting of the national council on radiation protection and measurements. No. 5, National Council on Radiation Protection and Measurements, Bethesda, MD; 1983: 40-82.

Robison, W.L.; Conrado, C.L.; Phillips, W.A. Enjebi Island dose assessment. Livermore, CA: Lawrence Livermore National Laboratory Report No.UCRL-53805; 1987.

Robison, W.L.; Conrado, C.L.; Stuart, M.L. Radiological conditions at Bikini Atoll: Radionuclide concentrations in vegetation, soil, animals, cistern water, and ground water. LIvermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-53840; 1988.

Robison, W.; Mount, M.; Phillips, W.; Conrado, C.; Stuart M.; Stoker, C. The northern Marshall Islands radiological survey: Terrestrial food chain and total doses. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-52833, Pt. 4: 1982a.

Robison, W.L.; Mount, M.E.; Phillips, W.A.; Stuart, M.L.; Thompson, S.E.; Conrado, C.L.; Stoker, A.C. An updated radiological dose assessment of Bikini and Eneu Islands at Bikini Atoll. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-53225; 1982b.

Robison, W.L.; Mount, M.E.; Phillips, W.A.; Stuart, M.L.; Thompson, S.E.; Conrado, C.L.; Stoker, A.C. An updated radiological dose assessment of Bikini and Eneu Islands at Bikini Atoll: Appendix A MLSC Ujelang Dietary Survey. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-53225 App. A.; 1982c.

Robison, W.L.; Phillips, W.A.; Mount, M.E.; Clegg, B.R.; Conrado, C.L. Reassessment of the potential radiological doses for residents resettling Enewetak Atoll. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-53066; 1980.

Robison, W. L.; Stone, E. L. The effect of potassium on the uptake of 137Cs in food crops grown in coral soils: Coconut at Bikini Atoll. Health Physics 62: 496-511; 1992.

Robison, W.L.; Phillips,W.A. Estimates of the radiological dose from ingestion of ^{137}Cs and ^{90}Sr to infants, children, and adults in the Marshall Islands. Livermore, CA: Lawrence Livermore National Laboratory Report No. UCRL-53917; 1989.

Schwarz, G.; Dunning, D. E., Jr. Imprecision in estimates of dose from ingested ^{137}Cs due to variability in human biological characteristics. Health Phys. 43:631–645; 1982.

Schwarz, G.; Hoffman, F.O. Imprecision of dose predictions for radionuclides released to the environment: An application of a Monte Carlo simulation technique. Environ. International 4:289-297; 1980.

Unnikrishnan, K.; Prasad, M. A. The lung model and fluctuations in air activity. Health Phys. 52:229–231; 1987.

Wang, M.C.; Uhlenbeck, G.E. On the theory of Brownian motion II. Rev. Mod. Phys. 17:323-342; 1945.

Whicker, F.W.; Kirchner, T. B. PATHWAY: A dynamic food-chain model to predict radionuclide ingestion after fallout deposition. Health Phys. 52:717–737; 1987.

Whicker, F.W.; Kirchner, T. B.; Breshears, D. D.; Otis, M. D. Estimation of radionuclide ingestion: The PATHWAY food-chain model. Health Phys. 59:645–657; 1990.

Appendix A. Properties of the metabolic transfer matrix

The matrix **A'** defined by ICRP(1979) is a lower triangular matrix

$$a_{ij} = 0 \quad \text{for } i < j \tag{A.1}$$

whose diagonal elements are in the form

$$a_{ii} = -\mu_i - \lambda \tag{A.2}$$

where μ_i is the biological turnover rate of the radionuclide in the ith compartment. Thus, it will be convenient to introduce a new matrix **A** of just the biological parameters defined by

$$\mathbf{A'} = \mathbf{A} - \lambda\mathbf{I} \tag{A.3}$$

where **I** is the identity matrix. The jth eigenvalues of **A'** is denoted by λ_j and is chosen so that the jth eigenvalue of **A** is $-\mu_j$. Kercher (1983) discusses the solution of linear transport models of the form of eqn. 1 using eigenvalues and eigenvectors. Let us assume that the eigenvalues of **A'** are discrete (not degenerate) so that the eigenvectors of **A'** are linearly independent. Denoting the jth eigenvector of **A'** as ψ^j, the eigen equation for **A'** is

$$\mathbf{A'}\,\psi^j = \lambda_j\,\psi^j \tag{A.4}$$

Eqn. A.1 implies that the eigenvalues of **A'** are the diagonal matrix elements so that

$$\lambda_i = -\mu_i - \lambda \tag{A.5}$$

which is substituted along with eqn. A.3 into eqn. A.4 to get

$$\mathbf{A}\,\psi^j = -\,\mu_j\,\psi^j \quad . \tag{A.6}$$

Thus the eigenvectors of **A** are the eigenvectors of **A'** and the eigenvalues of **A** are $-\mu_i$. Eqn. A.4 can be written as

$$\mathbf{A}'\,\Psi = \Psi\,\Lambda \tag{A.7}$$

where Ψ is a matrix with elements $\Psi_{ij} = \psi^j_i$ and Λ is the diagonal matrix with diagonal elements $\Lambda_{ii} = \lambda_i$. Note that $\Lambda_{ij} = 0$ for $i \neq j$. Because the eigenvectors of A' are linearly independent, Ψ^{-1} exists and eqn. A.7 implies

$$\Psi^{-1}\,\mathbf{A}'\,\Psi = \Lambda \tag{A.8}$$

and in fact

$$\Psi^{-1}\,f(\mathbf{A}')\,\Psi = f(\Lambda) \tag{A.9}$$

and for A one finds

$$\Psi^{-1}\,\mathbf{A}\,\Psi = \Omega \tag{A.10}$$

where Ω is the diagonal matrix with diagonal elements $\Omega_{jj} = -\,\mu_j$.and $\Omega_{ij} = 0$ for $i \neq j$.

Appendix B. Properties of uncertainty distributions

In this paper, use the usual functions to characterize the distribution of random variables, namely, the mean and standard deviation. For a random variable X, take the usual definitions of the expectation function $E(X)$ or mean of X (Cramer 1955) such that $E(X)$ has the properties

$$E(aX + b) = a\,E(X) + b \tag{B.1}$$

and

$$E\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i E(X_i) \tag{B.2}$$

where a and b are scalars. The standard deviation function $D(X)$ is given by

$$D^2(X) = \sigma^2 = E\left((X - E(X))^2\right) = E(X^2) - E^2(X) \tag{B.3}$$

with the property

$$D(aX + b) = |a|\, D(X)\ . \tag{B.4}$$

Use the result that if the random variables X_i are independent then

$$E\left(\prod_{i=1}^{k} X_i\right) = \prod_{i=1}^{k} E(X_i) \tag{B.5}$$

and

$$D^2\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i^2\, D^2(X_i)\ . \tag{B.6}$$

To derive eqn. B.6, one uses eqn. B.5. Furthermore, for X and Y independent random variables and using eqns. B.3 and B.5, one finds that

$$D^2(X\ Y) = D^2(X)\, D^2(Y) + E^2(X)\, D^2(Y) + E^2(Y) D^2(X)\ . \tag{B.7}$$

Appendix C. Calculations for the Random case

To characterize the distribution of y_k, take the expectation of eqn. 17 to get

$$E(y_k(T)) = e^{-\lambda T}\left[e^{\mathbf{A}T}\left(-\mathbf{A}^{-1}\right)\left(\mathbf{I} - e^{\mathbf{A}\Delta}\right) \sum_{i=1}^{N} E(C_i)\ e^{-\mathbf{A}i\Delta}\ \mathbf{G} \right]_k . \qquad \text{(C.1)}$$

Assume that each sampling is independent but from the same distribution (nonseasonal), hence $E(C_i) = c_R$ where c_R is the mean soil concentration with radioactive decay removed. So eqn, C.1 is simplified to

$$E(y_k(T)) = e^{-\lambda T}\left[e^{\mathbf{A}T}\left(-\mathbf{A}^{-1}\right)\left(\mathbf{I} - e^{\mathbf{A}\Delta}\right) c_R \sum_{i=1}^{N} e^{-\mathbf{A}i\Delta}\ \mathbf{G} \right]_k \qquad \text{(C.2a)}$$

$$= e^{-\lambda T}\left[e^{\mathbf{A}T}\left(-\mathbf{A}^{-1}\right)\left(\mathbf{I} - e^{\mathbf{A}\Delta}\right) c_R \left(e^{-\mathbf{A}\Delta} - e^{-\mathbf{A}\Delta(N+1)}\right)\left(\mathbf{I} - e^{-\mathbf{A}\Delta}\right)^{-1}\ \mathbf{G} \right]_k . \qquad \text{(C.2b)}$$

This expression simplifies to eqn. 18 in the text, i.e.,

$$E(y_k(T)) = c_R\, W_k(T) \qquad \text{(C.2c)}$$

where

$$W_k(T) = e^{-\lambda T}\left[\mathbf{A}^{-1}\left(e^{\mathbf{A}T} - \mathbf{I}\right)\mathbf{G}\right]_k . \qquad \text{(C.2d)}$$

Now consider the standard deviation for the Random case. Begin with eqn. 7b and employ the Random case assumptions outlined above so that eqn. 7b becomes

$$y_k(T) = \sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T} \sum_{i=1}^{N} C_i\, \Psi_{jl}^{-1}\, G_l \int_{(l-1)\Delta}^{l\Delta} e^{-(\lambda_j + \lambda)t}\ dt \qquad \text{(C.3a)}$$

$$= \sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T}\, \Psi_{jl}^{-1}\, G_l\, \frac{1 - e^{(\lambda_j + \lambda)\Delta}}{\left(-\lambda_j - \lambda\right)} \sum_{i=1}^{N} C_i\, e^{-(\lambda_j + \lambda)\Delta i} . \qquad \text{(C.3b)}$$

Interchange the summations in eqn. C.3b and take D^2 of both sides of the equation. Then use eqn. B.6 because the C_i are independent. One finds

$$D^2(y_k) = \sum_{i=1}^{N} D^2(C_i) \left[\sum_{j,l=1}^{n} \Psi_{kj}\ e^{\lambda_j T} \Psi_{jl}^{-1} G_l \frac{\left(1 - e^{(\lambda_j + \lambda)\Delta}\right)}{\left(-\lambda_j - \lambda\right)} e^{-(\lambda_j + \lambda)\Delta i} \right]^2 . \qquad \text{(C.4a)}$$

Using $D^2(C_i) = \sigma_R^2$, perform the sum over i in eqn. C.4a, and also use eqn. A.5 to find

$$D^2(y_k(T)) = \sigma_R^2\ S_k^2 \qquad \text{(C.4b)}$$

where

$$S_k^2 = e^{-2\lambda T} \sum_{j,l,m,r=1}^{n} \Psi_{kj}\ \Psi_{km}\ e^{-(\mu_j + \mu_m)T}\ \Psi_{jl}^{-1}\ \Psi_{mr}^{-1}\ G_l\ G_r$$
$$\left\{ \frac{\left(1 - e^{-\mu_j \Delta}\right)\left(1 - e^{-\mu_m \Delta}\right)}{\mu_j\ \mu_m} \frac{\left(1 - e^{(\mu_j + \mu_m)T}\right)}{\left(e^{-(\mu_j + \mu_m)\Delta} - 1\right)} \right\} . \qquad \text{(C.4c)}$$

Rapid Metabolism limit

Let us first consider the variance of the body burdens. In the Rapid Metabolism limit, all μ_i are large such that $\mu_i\ \Delta >> 1$. In this limit, the expression in the curly brackets in eqn. C.4c approaches

$$\{\ldots\}_{\text{eq. C.4c}} \rightarrow \left\{ \mu_j^{-1}\ \mu_m^{-1}\ e^{(\mu_j + \mu_m)T} \right\} \qquad \text{(C.5)}$$

so that eqn C.4b approaches

$$D^2(y_k) \rightarrow \sigma_R^2\, e^{-2\lambda T} \sum_{j,l=1}^{n} \left[\Psi_{kj}\, \mu_j^{-1}\ \Psi_{jl}^{-1}\ G_l \right] \sum_{m,r=1}^{n} \left[\Psi_{km}\, \mu_m^{-1}\ \Psi_{mr}^{-1}\ G_r \right] \qquad \text{(C.6a)}$$
$$\rightarrow \sigma_R^2\, e^{-2\lambda T} \left[-\mathbf{A}^{-1}\ \mathbf{G} \right]_k^2 . \qquad \text{(C.6b)}$$

The square root of eqn. C.6b is shown in the text, eqn. 25.

Slow Metabolism limit

Now consider the limit of Slow Metabolism. In this limit, all the μ_i are small so $\mu_i T << 1$. First, let us examine the uncertainty in the body burdens. In this limit, eqn. C.4b goes to

$$D^2(y_k) \rightarrow \sigma_R^2 e^{-2\lambda T} \sum_{j,l,m,r=1}^{n} \Psi_{kj} \Psi_{km} e^{-(\mu_j+\mu_m)T} \Psi_{jl}^{-1} \Psi_{mr}^{-1} G_l G_r \Delta^2 \frac{T}{\Delta} \quad \text{(C.7a)}$$

$$\rightarrow \sigma_R^2 e^{-2\lambda T} \frac{T^2}{N} G_k^2 \ . \quad \text{(C.7b)}$$

The square root of eqn. C.7b is shown in the text as eqn. 27. Note that eqn. 27 and 28 imply that to force $D(y_{k,Constant\ Input})$ to be equal to $D(y_{k,Random})$ requires that one set σ_{CI} to $\sigma_R/\sqrt{N}$.

Consider next the uncertainty in the dose. In addition to the Slow Metabolism limit, assume that $\lambda T << 1$. In this limit, the variance of the dose in the Random case (eqn. 22) approaches

$$D^2(H_k) \rightarrow \sigma_R^2 \sum_{j,l,h,p,m,q=1}^{n} B_{kj} \Psi_{jl} B_{kh} \Psi_{hp} \Psi_{lm}^{-1} G_m \Psi_{pq}^{-1} G_q \frac{\Delta^4 N(N-1)(N+1)}{3} \quad \text{(C.8a)}$$

$$\rightarrow \sigma_R^2 [\mathbf{B}\ \mathbf{G}]_k^2 \frac{T^4}{3N} \ . \quad \text{(C.8b)}$$

The square root of eqn. C.8b is given in eqn. 30. Eqn. 31 implies that to force $D(H_{k,Constant\ Input})$ to equal $D(H_{k,Random})$ requires that one set σ_{CI} to $2\sigma_R/(3N)^{1/2}$. Note that one can not force equality for both body burden and dose, simultaneously.

Appendix D. Autocorrelation in the Autoregressive model

To give meaning to the parameter α introduced in eqn. 34b, consider the autocorrelation function defined for a random variable X_t as

$$\rho(t,\tau) = \frac{Cov(X_t, X_{t+\tau})}{Var(X_t)} = \frac{E[(X_t - E(X_t))(X_{t+\tau} - E(X_{t+\tau}))]}{E[(X_t - E(X_t))(X_t - E(X_t))]} \quad . \tag{D.1a}$$

So, the autocorrelation function of the C_i defined by eqn. 34b is found by substituting eqn. 35 into the definitions of the covariance and variance. For i>1

$$\begin{aligned} Cov(C_i, C_{i+k}) = E\Bigg[& \left\{ \alpha^{i-1}(C_1 - c_A) + \sum_{j=2}^{i} \alpha^{i-j}(\varepsilon_j - \varepsilon_m) \right\} \\ & \left\{ \alpha^{i-1+k}(C_1 - c_A) + \sum_{j=2}^{i} \alpha^{i+k-j}(\varepsilon_j - \varepsilon_m) \right\} \Bigg] \quad . \end{aligned} \tag{D.1b}$$

Now note that since C_1 is independent of ε_i, use eqn. B.5 to find

$$E[(C_1 - c_A)(\varepsilon_j - \varepsilon_m)] = E(C_1 - c_A)\, E(\varepsilon_j - \varepsilon_m) = 0 \quad . \tag{D.2}$$

So that for $i > 1$

$$Cov(C_i, C_{i+k}) = \alpha^{2i-2+k} E[(C_1 - c_A)^2] + \sum_{j=2}^{i} \alpha^{i-j}\, \alpha^{i+k-j}\, E[(\varepsilon_j - \varepsilon_m)^2] \tag{D.3a}$$

$$= \alpha^{2i-2+k}\, \sigma_A^2 + \sigma_\varepsilon^2\, \alpha^{2i+k} \sum_{j=2}^{i} \alpha^{-2j} \tag{D.3b}$$

$$= \alpha^{2i-2+k} \left[\sigma_A^2 + \sigma_\varepsilon^2 \frac{\alpha^{-2(i-1)} - 1}{1 - \alpha^2} \right] \quad . \tag{D.3c}$$

Following the same procedure, the variance for $i > 1$ is given by

$$Var(C_i) = \alpha^{2i-2} \left[\sigma_A^2 + \sigma_\varepsilon^2 \frac{\alpha^{-2(i-1)} - 1}{1 - \alpha^2} \right] \quad . \tag{D.4}$$

So

$$\rho(i,k)=\frac{\alpha^{2i-2+k}\left[\sigma_A^2+\sigma_\varepsilon^2\dfrac{\alpha^{-2(i-1)}-1}{1-\alpha^2}\right]}{\alpha^{2i-2}\left[\sigma_A^2+\sigma_\varepsilon^2\dfrac{\alpha^{-2(i-1)}-1}{1-\alpha^2}\right]}=\alpha^k \qquad i>1\ . \tag{D.5}$$

The derivation for $i = 1$ is even simpler

$$\mathrm{Cov}(C_1,C_{1+k})=E\left[(C_1-c_A)\left\{\alpha^k(C_1-c_A)+\sum_{j=2}^{1+k}\alpha^{1+k-j}(\varepsilon_j-\varepsilon_m)\right\}\right] \tag{D.6a}$$

$$=\alpha^k\ E[(C_1-c_A)^2]=\alpha^k\ \sigma_A^2 \tag{D.6b}$$

and

$$\mathrm{Var}(C_1)=\sigma_A^2\ . \tag{D.6c}$$

So

$$\rho(1,k)=\alpha^k\ . \tag{D.7}$$

Therefore, for all i and k one finds

$$\rho(i,k)=\alpha^k \qquad \textit{for}\ i\geq 1,\ k\geq 0\ . \tag{D.8}$$

Appendix E. Calculations in the Autoregressive case

To calculate the uncertainty in the distribution of y_k in the Autoregressive case, begin with eqn. C.3b, which is rewritten as

$$y_k(T)=\sum_{j,l=1}^{n}\Psi_{kj}\,e^{\lambda_j T}\,\Psi_{jl}^{-1}\,G_l\frac{1-e^{-\mu_j\Delta}}{\mu_j}\left[C_1\,e^{\mu_j\Delta}+\sum_{i=2}^{N}C_i\,e^{\mu_j i\Delta}\right] \tag{E.1}$$

and substitute eqn. 35 for C_i to get

$$y_k(T) = \sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T}\, \Psi_{jl}^{-1}\, G_l \frac{1 - e^{-\mu_j \Delta}}{\mu_j} \left[C_1\, e^{\mu_j \Delta} + \sum_{i=2}^{N} \left(\alpha^{i-1} C_1 + \sum_{r=2}^{i} \alpha^{i-r}\, \varepsilon_r \right) e^{\mu_j i \Delta} \right] . \qquad \text{(E.2)}$$

Then simplify by carefully interchanging summations and summing over i to get

$$y_k(T) = \sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T}\, \Psi_{jl}^{-1}\, G_l \frac{1 - e^{-\mu_j \Delta}}{\mu_j} \left[C_1\, e^{\mu_j \Delta} \frac{1 - \alpha^N e^{\mu_j T}}{1 - \alpha\, e^{\mu_j \Delta}} + \sum_{r=2}^{N} \varepsilon_r\, e^{\mu_j r \Delta} \frac{1 - \alpha^{N-r+1}\, e^{\mu_j \Delta (N-r+1)}}{1 - \alpha\, e^{\mu_j \Delta}} \right] . \qquad \text{(E.3)}$$

Take D^2 of both sides of eqn. E.3 and since all the random variables in eqn. E.3 are independent, apply eqn. B.6 to get

$$D^2(y_k) = \left[\sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T}\, \Psi_{jl}^{-1}\, G_l \frac{1 - e^{-\mu_j \Delta}}{\mu_j}\, e^{\mu_j \Delta} \frac{1 - \alpha^N e^{\mu_j T}}{1 - \alpha\, e^{\mu_j \Delta}} \right]^2 D^2(C_1) + \sum_{i=2}^{N} D^2(\varepsilon_i) \left[\sum_{j,l=1}^{n} \Psi_{kj}\, e^{\lambda_j T}\, \Psi_{jl}^{-1}\, G_l \frac{1 - e^{-\mu_j \Delta}}{\mu_j}\, e^{\mu_j i \Delta} \frac{1 - \alpha^{N-i+1}\, e^{\mu_j \Delta (N-i+1)}}{1 - \alpha\, e^{\mu_j \Delta}} \right]^2 . \qquad \text{(E.4)}$$

Using $D^2(C_1) = \sigma_A^2$ and $D^2(\varepsilon_i) = \sigma_\varepsilon^2$, perform the sum over i and this equation becomes eqn. 39.

Rapid Metabolism limit

In the limit of Rapid Metabolism, the expectation value of the body burden y_k for the Constant Input case, eqn. 11, approaches

$$E(y_{k,\ \text{Constant Input}}) \rightarrow e^{-\lambda T}\, c_{CI} \left[-\mathbf{A}^{-1}\, \mathbf{G} \right]_k \qquad \text{(E.5)}$$

whereas the expectation value for the Autoregressive case goes to

$$E(y_{k,\ Autoregressive}) \rightarrow \left[\alpha^{N-1} + \frac{\varepsilon_m}{c_A(1-\alpha)}\left(1-\alpha^{N-1}\right)\right] e^{-\lambda T} c_A \left[-\mathbf{A}^{-1}\ \mathbf{G}\right]_k \quad . \qquad \text{(E.6)}$$

These two equations are combined to produce eqn. 43.

Now consider the uncertainty of the body burden in the Rapid Metabolism limit. In this limit, as before, the uncertainty of the Constant Input case (eqn. 12) approaches eqn. 19 with σ_{CI} substituted for σ_R, and the uncertainty in the Autoregressive case approaches

$$D^2(y_{k,\ Autoregressive}) \rightarrow e^{-2\lambda T}\,\sigma_A^2 \left[\alpha^{2(N-1)} + \frac{\sigma_\varepsilon^2\left(1-\alpha^{2(N-1)}\right)}{\sigma_A^2\left(1-\alpha^2\right)}\right]\left[-\mathbf{A}^{-1}\ \mathbf{G}\right]_k^2 \quad . \qquad \text{(E.7)}$$

Combining these two results one arrives at eqn. 44 in the text.

To calculate the expectation of the dose H_k in the Rapid Metabolism limit, assume that the decay is rapid, too, i.e., assume $\lambda\Delta >> 1$. Then the expectation of the dose in the Autoregressive case approaches eqn. 26 if c_A is substituted for σ_{CI} and the expectation of the dose in the Constant Input case in the same limit also approaches eqn. 26 if c_{CI} is substituted for σ_{CI}. These results combine to produce equality of $E(H_{k,Autoregressive})$ and $E(H_{k,Random})$ for $c_{CI}=c_A$.

Now consider the uncertainty in the dose for the Rapid Metabolism limit. In this limit, eqn. 42a approaches

$$D^2(H_{k,\ Autoregressive}) \rightarrow \frac{\sigma_A^2}{\lambda^2}\left[\mathbf{B}\left(-\mathbf{A}+\lambda\,\mathbf{I}\right)^{-1}\ \mathbf{G}\right]_k^2 \qquad \text{(E.8)}$$

just as the Constant Input case , eqn. 15, approaches eqn. 26. These two equations show equality of $D(H_{k,Constant\ Input})$ and $D(H_{k,Autoregressive})$ in the Rapid Metabolism limit for $\sigma_A=\sigma_{CI}$.

Slow Metabolism limit

In the limit of Slow Metabolism for the radionuclide in question, the expectation value of the body burden y_k for Constant Input case, eqn. 11, approaches eqn. 28 if one replaces σ_{CI} by c_{CI} and the expectation value of the body burden in the Autoregressive case, eqn. 37, approaches

$$E(y_{k,\ Autoregressive}) \rightarrow \left[\frac{\varepsilon_m}{c_A(1-\alpha)} + \frac{1}{N}\frac{1-\alpha^N}{1-\alpha}\left(1 - \frac{\varepsilon_m}{c_A(1-\alpha)}\right)\right] e^{-\lambda T}\, c_A\, T\, G_k \quad . \text{(E.9)}$$

Use these results to get eqn. 45.

To calculate the uncertainty in the body burden y_k in the Slow Metabolism limit for the Constant Input case take the limit of eqn. 12 to find eqn. 28, as before, and the limit for the Autoregressive case, eqn. 39 becomes

$$D^2(y_{k,\ Autoregressive}) \rightarrow \left[\frac{1}{N^2}\left(\frac{1-\alpha^N}{1-\alpha}\right)^2 + \frac{\sigma_\varepsilon^2}{\sigma_A^2 N^2 (1-\alpha)^2}\left\{N - 1 - \left(\frac{1-\alpha^N}{1-\alpha^2}\right)(1 - \alpha^N + 2\alpha)\right\}\right] \sigma_A^2\, e^{-2\lambda t}\, T^2\, G_k^2 \quad . \text{(E.10)}$$

These two results can be combined to produce eqn. 46 in the text. Note that in the limit of large N, under stationary conditions, to force $D^2(y_{k,Constant\ Input})$ to equal $D^2(y_{k,Autoregressive})$ requires that σ_{CI} be set to $\sigma_A\{(1+\alpha)/[N(1-\alpha)]\}^{1/2}$.

The expectation of the dose for the Constant Input case in the Slow Metabolism (and decay) limit approaches

$$E(H_{k,\ Constant\ Input}) \rightarrow c_{CI}[\mathbf{B}\ \mathbf{G}]_k \frac{T^2}{2} \quad \text{(E.11)}$$

and the expectation of the dose in the Autoregressive case approaches the same expression if $\varepsilon_m/(1-\alpha)$ is substituted for c_{CI}. Use these two results to produce eqn. 48.

Recall that the uncertainty of the dose for the Constant Input case in the Slow Metabolism (and decay) limit, eqn. 15 approaches eqn. 31 and the uncertainty of the dose for the Autoregresseive case , eqn. 42a, in this limit approaches

$$D^2(H_{k,\ Autoregressive}) \rightarrow \frac{\sigma_\varepsilon^2}{(1-\alpha)^2}[\mathbf{B}\ \mathbf{G}]_k^2 \frac{T^4}{3N} \ . \qquad \text{(E.12)}$$

Combining these results, one gets eqn. 49a which relates the uncertainties of the two doses. To force the variance of $H_{k,Constant\ Input}$ to equal that of $H_{k,Autoregressive}$ requires that one sets σ_{CI} to $\sigma_A\{4(1+\alpha)/[3N(1-\alpha)]\}^{1/2}$. Note that this differs from the result for the body burden by a factor of $2/3^{1/2}$. Thus to adjust σ_{CI} to force the uncertainty of Constant Input dose to match the uncertainty of the Autoregressive dose, the uncertainties of the body burdens will differ.

Table 1. Glossary of symbols with the first equation in which the symbol appears and a brief description of the symbol.

Symbol	First Eqn.	Description
a_{ij}	1	Constant coefficient of transfer from compartment j to i (d^{-1}) for j different from i. For j and i the same, negative of the sum of radioactive decay rate and total elimination rate from compartment i.
A	7a	Matrix of biological turnover rates and transfer coefficients (d^{-1}). Matrix **A** is matrix **A'** with radioactive decay rate removed.
A'	3	Matrix of matrix elements a_{ij}
α	35b	Correlation of soil exposure C_i to C_{i+1} in Autoregressive case
α_i	34	Autocorrelation coefficient of X_t and X_{t-i} in general linear model of autocorrelation
B_{kj}	2	Conversion factor of burden-to-dose that is proportional to SEE coefficients for the kth target organ by the jth source organ
B	13a	Matrix of B_{ij} matrix elements
β_i	34	Correlation coefficient of X_t and Z_{t-i} in general model of autocorrelation
c_A	38	Mean soil concentration of initial exposure in Autoregressive case (Bq cm^{-3})
c_{HC}	11	Mean soil concentration for initial exposure in Constant Input case (Bq cm^{-3})
c_R	15	Mean soil concentration for exposures in Random case (Bq cm^{-3})
c_s	5	Soil concentration of radionuclide (Bq cm^{-3})
C_i	14	Random variable of soil concentration during exposure period i in Random and Autoregressive cases (Bq cm^{-3})
C_s	5	Random variable of soil concentration with dependence on radioactive decay removed (Bq cm^{-3})
D	12	Standard deviation function

Δ	14	Length of time subintervals for constant exposures in Random and Autoregressive cases (d)
E	11	Expectation function
ε_i	35b	Random or uncorrelated stochastic portion of soil exposure C_i in Autoregressive case (Bq cm^{-3}) or random portion of daily intake of ^{137}Cs in Autoregressive case of Bikini Island example (Bq d^{-1})
ε_m	38	Mean of random or uncorrelated portion of soil exposure in Autoregressive case (Bq cm^{-3}) or mean of random portion of daily intake of ^{137}Cs in Bikini Island example (Bq d^{-1}).
f	A.9	Arbitrary function of matrix **A** that is transformed to same function of eigenvalue matrix under the similarity transformation on **A** performed by the eigenvector matrix
F_i	1	Intake (inhalation or ingestion) of radionuclide to compartment i (Bq d^{-1})
F	3	Vector of n components of F_i
G_i	5	Transfer function that gives radionuclide delivered to ith body organ from soil concentration (Bq(intake) d^{-1} Bq(soil)$^{-1}$ cm^3)
G	7a	Vector of G_i components
H_k	2	Cumulative dose to kth organ (Sv)
H	40a	Vector form of cumulative dose with vector element H_k
I	10	Identity matrix
J(t)	62	Total ingestion rate of ^{137}Cs (Bq d^{-1}) at time t in Bikini example.
$K_j(t)$	62	Concentration of ^{137}Cs (Bq g^{-1}) in foodstuff j at time t
λ	6	Radioactive decay rate (d^{-1})
λ_i	7b	Eigenvalue i of matrix **A'** (d^{-1}) and the negative of the sum of the radioactive decay rate and biological turnover rate of compartment i
Λ	A.7	Diagonal matirx of eigenvalues λ_i

M	40a	Function of scalar or matrix used for repetitive patterns to simplify complicated expression
μ_i	16	Biological turnover rate of compartment i and negative of eigenvalue i of matrix **A** (d^{-1})
n	1	Number of body compartments
N	14	Number of exposure intervals in Random and Autoregressive cases
N_4	52	Number of exposure intervals Δ in residence time τ_4 for Bikini example.
Ω	A.10	Diagonal matrix with diagonal elements μ_i
P	42a	Function of scalar used for repetitive patterns to simplify complicated expressions
Ψ_{kj}	7b	Matrix element given by the kth element of the jth eigenvector of matrices **A** and **A'**
Ψ	A.7	Matrix of eigenvectors with matrix elements Ψ_{ij}
$Q_j(t)$	62	Quantity of foodstuff j consumed per day at time t ($g\ d^{-1}$)
R	42a	Function of scalar used for repetitive patterns to simplify complicated expressions
$\rho(i,k)$	D.1a	Correlation coefficient of C_i and C_{i+k}
S_k^2	C.4c	Variance of body burden of compartment k at time T in Random case divided by $\sigma_R 2$.
σ_A	39	Standard deviation of soil concentration in initial exposure in the Autoregressive case (Bq cm^{-3})
σ_ϵ	39	Standard deviation of random or uncorrelated portion of soil exposure in the Autoregressive case (Bq cm^{-3})
σ_{HC}	12	Standard deviation of soil concentration for initial exposure in Constant Input case (Bq cm^{-3})
σ_R	16	Standard deviation of soil concentration in Random case (Bq cm^{-3})
t	1	Time variable (d)

T	2	Total period of exposure for cumulative dose (d)
τ_i	52	Residence time of radionuclide in compartment i (d).
W_k	C.2d	Expectation of body burden for compartment k at time T in Random case divided by c_R
X_t	34	Dependent random variable in general linear model of autocorrelation
y_i	1	Body burden of ith body organ or component (Bq)
$\mathbf{y}$	3	Vector of n components of y_i
Z_t	34	Independent random variable in uncorrelated stochastic portion of random variable X_t in general linear model of autocorrelation

Table 2. Statistics of measured concentrations of Bikini Island foodstuffs. Statistics of diet survey results of Marshallese women age 18 to 78.

Foodstuff j	Mean of Q_j (g d^{-1})	Standard Deviation of Q_j (g d^{-1})	Mean of K_j (Bq g^{-1})	Standard Deviation of K_j (Bq g^{-1})
Breadfruit	27	38	0.59	0.33
Cake from coconut[a]	12	8.6		
Coconut juice	99	98	1.6	1.3
Coconut milk[a]	52	66		
Copra	12	27	8.2	5.3
Drink. coconut meat[b]	32	65		
Sprouted coconut[a]	7.8	22		
Pandanus	8.7	17	6.7	5.4
Pork	5.7	10	8.5	6.6

[a]Use the same K statistics as copra

[b]Statistics of K are 2.67 times those of coconut juice for Bikini Island.

Figure Captions

Figure 1. Calculated mean and uncertainties (standard deviations) of total body burden of ^{137}Cs for women on Bikini Island. January 1, 1987 corresponds to t=0. Mean for all cases is the same. Four different cases of exposure are Constant Input Following Initial Random Exposure, Random, Mixed Random and Constant Input, and Autoregressive with α=0.5.

Figure 2. Uncertainty (standard deviation) of the total body burden of ^{137}Cs in the Autoregressive case for women on Bikini Island after three years of exposure. The uncertainty is plotted as a function of the daily autocorrelation (α) of total daily ingestion rate of ^{137}Cs. Note that as α approaches 0, the uncertainty approaches that for the Random case; and as α approaches 1, the uncertainty approaches that for the Constant Input case. Also, note the rapid rise of of the uncertainty for α greater than 0.9.

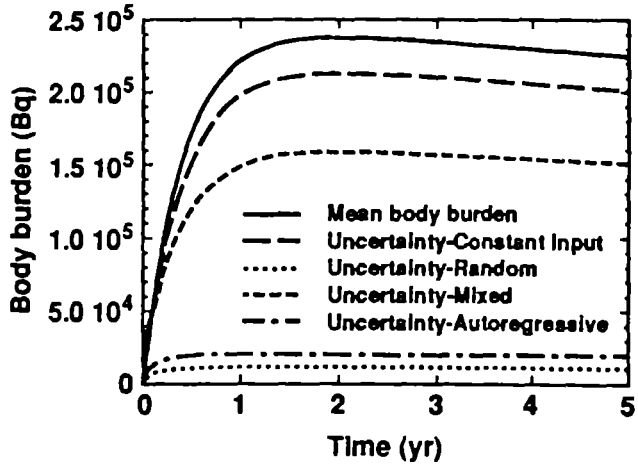
2.5 10^5
2.0 10^5
1.5 10^5
1.0 10^5
5.0 10^4
0
Body burden (Bq)
0
1
2
3
4
5
Time (yr)
Mean body burden
Uncertainty-Constant Input
Uncertainty-Random
Uncertainty-Mixed
Uncertainty-Autoregressive

Figure 1

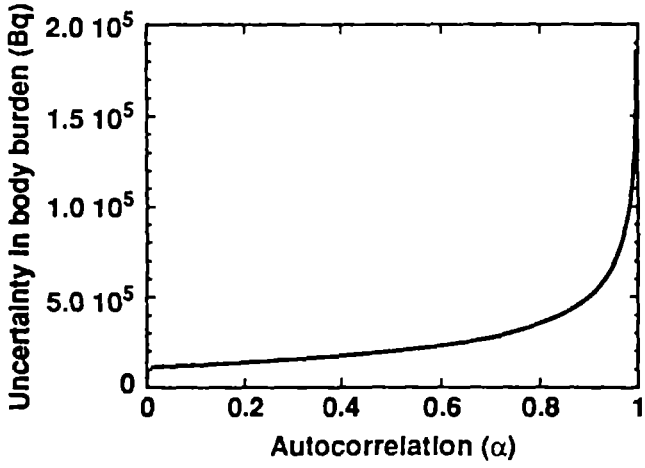
Uncertainty in body burden (Bq)
2.0 10^5
1.5 10^5
1.0 10^5
5.0 10^5
0
0
0.2
0.4
0.6
0.8
1
Autocorrelation (α)

Figure 2